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On Simulation of the Young Measures – Comparison of Random-Number Generators

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Abstract

"Young measure" is an abstract notion from mathematical measure theory. Originally, the notion appeared in the context of some variational problems related to the analysis of sequences of “fast” oscillating functions. From the formal point of view the Young measure may be treated as a continuous linear functional defined on the space of Carathéodory integrands satisfying certain regularity conditions. Calculating an explicit form of specific Young measure is a very important task. However, from a strictly mathematical standpoint it is a very difficult problem not solved as yet in general. Even more difficult would be the problem of calculating Lebasque’s integrals with respect to such measures. On the other hand in many real-world applications it would be enough to learn only some of the most important probabilistic characteristics of the Young distribution or learn only approximate values of the appropriate integrals. In such a case a possible solution is to adopt Monte Carlo techniques. In the paper we propose three different algorithms designed for simulating random variables distributed according to the Young measures associated with piecewise functions. Next with the help of computer simulations we compare their statistical performance via some benchmarking problems. In this study we focus on the accurateness of the distribution of the generated sample.

Keywords: Young measure, random numbers, piecewise functions, simulations.

1. Mathematical Background of the Young Measures

The very roots of the idea of the Young measures are in the twentieth problem in the famous list of twenty three open mathematical problems proposed by David Hilbert in the last year of the XIX century. The problem was: *Do all variational problems* *with certain boundary conditions have solutions?* The following problem is attributed to Oscar Bolza: minimize the functional function *u*:

$$J(u) := \int_{(0,1)} \left(u^2 + \left(\frac{du}{dx} \right)^2 - 1 \right)^2 dx,$$

subject to $u(0)=0=u(1)$.

The minimizing sequence for the functional J is of the following form:

$$u_n(x) := \begin{cases} x - \frac{k}{n}, & x \in \left(\frac{k}{n}, \frac{2k+1}{2n} \right) \\ -x + \frac{k+1}{n}, & x \in \left(\frac{2k+1}{2n}, \frac{k+1}{n} \right) \end{cases}$$

It can be seen that these elements oscillate more and more 'wildly' as the indices of the elements of this sequence grow.

The infimum of J is zero, but there is no function vanishing almost everywhere (with respect to the Lebesgue measure) in [0,1] with the derivative equal at the same time ±1 almost everywhere in this interval. Therefore J does not attain its infimum. Such functionals are often met in nonlinear elasticity, where minimization of the energy concerning multiple-well (i.e. *nonconvex*) potentials is considered. We often say that this is where *microstructure* appears; microstructure is then associated with the (not necessarily attained) minimizers of the functional J.

We can look at this problem from a bit more general point of view. Consider a sequence (u_n) of bounded functions defined on an open subset $\Omega \subset \mathbb{R}^n$ with positive Lebesgue measure and taking values in a compact subset $K \subset \mathbb{R}^m$. Assume further, that these functions belong to a suitable function space (e.g. L^∞) and that the sequence is weakly* convergent to u_0 while divergent in the norm topology. Moreover, let the elements of this sequence be oscillating functions, oscillating more and more rapidly around the weak* limit. It can be shown, that for any continuous function φ the weak* limit of the sequence $(\varphi(u_n))$ *is not* equal to $\varphi(u_0)$ unless φ is affine. It was Laurence Chisolm Young who in 1937, while considering nonconvex variational problem of that type, proposed to enlarge the space of admissible limits to the objects which he called 'generalized curves' (see [1]). These generalized curves are today commonly called 'Young measures' (other names are 'relaxed trajectories', 'transition probabilities').

More precisely, Young proved that there exists a subsequence of $(\varphi(u_n))$, not relabelled, and a family $(\nu_x)_{x\in\Omega}$ with support in K, such that for any continuous function φ on $\mathbb{R}^m$ and any function $g \in L^1(\Omega)$ there holds

$$\lim_{n\to\infty} \int_{\Omega} \varphi\big(u_n(x)\big) g(x) dx = \int_{\Omega} \int_{K} \varphi(x) \nu_x(ds)\, g(x) dx$$

$$:= \int_{\Omega} \bar{\varphi}(x) g(x) dx \tag{1}$$

It is seen, that the Young measure is indeed a generalized limit of the sequence under consideration (analogically, as in calculus a non-continuous function can be understood as a generalized limit of a sequence of continuous functions).

Today Young measures are widely used in many areas of pure and applied mathematics, engineering and even economics. There are also monographs devoted entirely or in part to them. For general theory the interested reader may consult e.g. the monograph [2] that is devoted entirely to the Young measures and treats this broad topic from general point of view. The title chapter starts with almost ten-page long part concerning abstract form of disintegration of measures. There is a whole chapter devoted to the Young measures on Banach spaces; the next one describes their applications in control theory, namely differential inclusions in Banach spaces are analyzed. Another interesting book [3] also provides the reader with abstract theoretical foundation of the Young measures, the title chapters deal exclusively with the Young measures in the topological framework of Souslin spaces. One may also find there, important in optimization, parts devoted to gradient Young measures. The monograph [4] is a classic book written by one of the most prominent authors in the field. Variational problems are treated with an emphasis on the quasiconvexity of integrands of minimized functionals. There is an entire chapter devoted to phase transition and microstructure, where the effects described by Young measures can be observed physically. In [5] a reader may find interesting facts related to optimization theory and variational calculus treated there in an abstract way. The third chapter of this work is crucial for our present article: Young measure is described there as an object associated with *any* measurable function, rather than a generalized limit generated by a proper sequence, like in [4]. For applications in differential equations and optimization the interested reader is referred to [6] – a massive treatise on contemporary calculus of variations with applications in PDE's, mechanics, computer vision, shape optimization, to mention some topics. Young measures basics are treated in the fourth chapter 'Complements on measure theory'. They are used in the sequel, especially in paragraphs investigating problems in nonlinear elasticity. Another important book devoted to the application of the Young measure theory is [7] where Young measures are used for investigating those evolution PDE's of the hyperbolic and parabolic type, which arise in nonlinear fluid mechanics. In this book the Young measures are introduced in the context of scalar conservation laws. There are also interesting articles devoted to engineering applications, see e.g. [8] or [9] as well as the numerical aspects of the usage of Young measures in differential equations and engineering, see e.g. [10] and [11]. References cited in these works lead to the more specialized topics.

As it has already been mentioned, Young measures can be looked at as generalized limits of the sequences $(\varphi(u_n))$, where (u_n) is a sequence of bounded functions and φ is a continuous function. Calculating weak* limit of $(\varphi(u_n))$ is rather difficult and usually requires some more advanced functional analytic tools. In the present work we will use another approach described in detail in [5]. It enables us to associate a Young measure with *any single* measurable function defined on the nonempty, open, bounded subset Ω of R with finite Lebesgue measure and values in the compact set $K \subset R$. Roughly speaking, having measurable function $u: \Omega \to K$, we consider the space $\mathcal{Y}$ of mappings ν defined on Ω with values being regular probability measures on *K*. We assume that each element $\nu \in \mathcal{Y}$ satisfies certain regularity conditions. This is the space of *all* Young measures. Next we imbed the function u in $\mathcal{Y}$ via Dirac mapping: we assign to any u the probability measure $\delta_{u(x)}, x \in \Omega$. It can be proved that there exists isometric isomorphism between the space $\mathcal{Y}$ and the space Y of the limits of kind described by (1). We refer the reader to [5] for detailed presentation of this approach.

It turns out that it is useful to introduce somewhat simpler objects than Young measures, namely the *quasi-Young measures.* Moreover, in many cases, important both from theoretical and practical points of view, the quasi-Young measures associated with functions under consideration, are exactly *the* Young measures associated with them, see [13].

The first advantage of this approach is the possibility of avoiding the use of advanced functional analytic tools while calculating their explicit form. The second advance is the possibility of the Monte Carlo simulating the (quasi-)Young measures associated with considered functions.

For the purpose of this presentation we will restrict ourselves to the case of piecewise affine functions of one real variable. More precisely, let $I \subset R$ be an open interval with Lebesgue measure μ normed to unity and let $\{I_i\}_{i=1}^{n}$ be an open partition of *I*. We will consider the functions that are affine on I_i, i.e. the functions of the form:

$$u(x) := \sum_{i=1}^{n} (a_i x + b_i)\chi_{I_i}(x)$$

with $a_i, b_i \in R$, where χ_A denotes the characteristic function of the set *A*. Moreover, assume that for any i=1,2,…, the closure of the set $\{a_i x + b_i : x \in I_i\}$ is compact. Let $K \subset R$ be a compact set with Lebesgue measure *dy*.

Definition 1. We say that a family of probability measures $\nu = (\nu_x)_{x \in I}$ is a quasi-Young measure associated with the measurable function $w: I \to K$ if for every continuous function $\beta: K \to R$ there holds an equality

$$\int_K \beta(k)\,d\nu_x(k) = \int_I \beta(w(x))\,d\mu \qquad (2)$$

In many cases the quasi-Young measure associated with the function under interest does not depend on the variable *x*. The family $\nu = (\nu_x)_{x \in I}$ of probability measures is then in fact a single measure ν and we say in this case that the (quasi)-Young measure is *homogeneous*. Quasi-Young measures associated with all the functions considered in this article are of this type.

Recalling that a function taking only finitely many values is called a *simple* (*or step*) *function,* definition 1 together with the change of variables theorem yields the following results (see [13]):

Theorem 1

(*a*) Young measure associated with a piecewise constant function, that is the function of the form $w(x)=\sum_{i=1}^{n} a_i \chi_{I_i}(x)$ is a convex combination of Dirac measures concentrated in the points a_i, $i=1,2,\ldots,n$. that is $\nu=(1/M)\sum_{i=1}^{n} m_i \delta_{a_i}$, where M, m_i are Lebesgue measures of the intervals I and I_i respectively;

(*b*) Young measure associated with a piecewise strictly monotonic affine function w on I, that is, $w(x)=\sum_{i=1}^{n}(a_i x+b_i)\chi_{I_i}(x):=\sum_{i=1}^{n} w_i(x)\chi_{I_i}(x)$ with $a_i \neq 0$ and $b_i \in R$ and additionally such that w is continuous on I with $\mathrm{cl}\,w_i(I_i)$ compact, is absolutely continuous with respect to the Lebesgue measure dy on K. In this case the Young measure is of the form

$$\nu=(1/M)\sum_{i=1}^{n}\left|1/a_i\right|\chi_{w_i(I_i)}dy.$$

Note that a function $w(.)$ of the form given in the point (*a*) of Theorem 1 is a *simple function*.

2. Generating of Young Measures Associated with Simple Functions

The idea underlying any Monte Carlo simulation is to draw a sample i.e. a realization of the stochastic process $\{Z_1, Z_2, \ldots Z_m\}$ composed of independent random variables with the same distribution as the random phenomenon under study. Based on this sample, important information concerning stochastic characteristics of the examined distribution can be derived with the help of statistical-inference tools. Indeed, by the strong law of large numbers, for any Borel function f for which the expected value $Ef(Z)$ exists, the average $\bar{f}_m=\frac{1}{m}\sum_{i=1}^{m} f(Z_i)$ will almost surely (a.s.) converge to $Ef(Z)$. In particular, when the sample size m tends to infinity, we can quite precisely evaluate all moments of the investigated distribution (e.g. expected value, variance etc.) as well as probabilities of related random events. The latter can be used for evaluating the theoretical frequencies of various intervals, so we can also obtain a histogram that approximates the distribution density function. The approximation of the density function is the better, the longer is the observation sequence.

Now let us consider simple functions f defined on a bounded interval I, and such that their values have inverse images being the intervals or their unions. By Theorem 1 (point (a)) the Young measures associated with such functions are the discrete probability distributions which can be easily simulated by computer procedures. In Monte Carlo simulations, the sample of random variables having such distributions can be generated according the following routine DYM(f , I, N).

```
Set k=1;
While n≤N Do Step 1 to Step 3
    Step 1. Set t=Random(I)
    Step 2. Set z[k]=f(t)
    Step 3. Set k=k+1
Set sample=(z[1],...,z[N])
Return sample
```

The procedure DYM is called with three arguments: the formula *f* that defines the simple function, its domain, i.e. the interval I, and the sample size N
The subroutine `Random`(I) returns a pseudorandom number generated according to the uniform probability distribution defined on I.

3. Generating of Young Measures Associated with Piecewise Affine Functions

The mathematical idea underlying the routine DYM can also be adopted to approximate Young measures in cases that are a bit more sophisticated. It is well known from the measure theory, that any Borel function can be approximated with the simple function (more precisely it is a limit of a sequence of the simple functions) Thus it can be expected, that for a large class of functions, we can approximate related Young measures by a properly chosen simple function. In this paper, such an approximation of any function *f* will be addressed to as *simple approximation*, and denoted as SA(*f*). Here we propose the following general steps of the construction of such a simple approximation for any piecewise function *f* determined on the interval *I*.
GS 1. Split the interval I=(a,b) into n equal in length subintervals $I_1,...,I_n$.
GS 2. Choose the sequence $Y=\{y_i\}$, where $y_i = f(x_i)$, $i=1,...,n$, and x_i is the *centre* of the subinterval I_i.
GS 3. As SA(*f*) choose the following simple function u:

$$u(x) := \sum_{k=1}^{n} y_k \chi_{I_k}(x)$$

Below we present a routine AYM1(f , a, b, N, n) that realizes the above general steps GS 1-3, and returns a sample from the (approximated) Young measure related to any piecewise function *f*.

```
Set k=1;
Set jump=(b-a)/n
While k≤n Do Step 1 to Step 3
```

```
    Step 1. Set t=a+(k-1/2)*jump
    Step 2. Set y[k]=f(t)
    Step 3. Set k=k+1
 Set i=1;
 While i≤N Do Step 4 to Step 6
    Step 4. Set k=RandomI({1,..,n})
    Step 5. Set z[i]=y(k)
    Step 6. Set i=i+1
 Set sample=(z[1],...,z[N])
 Return sample
```

The arguments for the AYM1 are the following: the formula f that defines the simple function, the endpoints of the interval $I=(a,b)$, the number n of subintervals $I_1,...,I_n$, and the sample size N.

The subroutine `RandomI`(A) returns a pseudorandom integer number generated according to the uniform probability distribution defined on A - a finite subset of integers. The procedure AYM1 was introduced in [14]. However it turns out, that, although rarely, in some cases the AYM1 fails to generate distributions that satisfy the benchmarking tests. Thus we propose here to modify this procedure by changing the GS2. Instead of choosing the sequence $y_i = f(x_i)$, $i=1,...,n$, where x_i is the *centre* of the subinterval I_i, we adopt the following definition: $y_i = f$ (`Random`(I_i)), $i=1,...,n$. Consequently, the only difference now is that the argument x_i is a random number belonging to I_i. As a result we get at the following routine AYM2:

```
 Set k=1;
 Set jump=(b-a)/n
 While k≤n Do Step 1 to Step 3
    Step 1. Set t=a+(k-Random([0,1])*jump
 Step 2. Set y[k]=f(t)
   Step 3. Set k=k+1
 Set i=1;
 While i≤N Do Step 4 to Step 6
   Step 4. Set k=RandomI({1,..,n})
   Step 5. Set z[i]=y(k)
   Step 6. Set i=i+1
 Set sample=(z[1],...,z[N])
 Return sample
```

Both introduced procedures for generating Young measures are based on the steps GS1-GS3 which allow as to construct a simple approximation of the function f. In the case of the AYM1 it is done deterministically whilst in the case of AYM2 the numbers $a_i=y_i$, $i=1,...,n$, necessary for the simple approximation are taken at random. As a result in both cases we end up with specific sequence $Y=\{y_i\}$, $i=1,...,n$, that "represents" the values of f taken on within each related interval I_i. One may ask however, why we should confine ourselves to this specific choice of the "representatives". It seems appealing to consider yet another version of the random-number generator. To describe the idea underlying this new routine let us note that in the AYM2 at first we draw a sequence *Y* and then, by drawing an interval (its number), we draw the representative for the value of f . It seems worth considering to change of the order of these two random drawings in a new generating procedure. Namely, in such a procedure at first we will draw the interval, and next the argument x_i belonging to this interval will be randomly chosen. Such idea is interesting because in such a way, we can significantly enrich the set of possible values *Y*, i.e. the set of the "representatives" of the f. Consequently, we propose to verify the usefulness the following computer routine AYM3:

```
Set k=1;
Set jump=(b-a)/n
While k≤N Do Step 1 to Step 4
  Step 1. Set k=RandomI({1,..,n})
  Step 2. Set t=a+(k-Random([0,1])*jump
  Step 3. Set z[k]=f(t)
  Step 4. Set k=k+1
Set sample=(z[1],...,z[N])
Return sample
```

Both new routines are called with the same arguments as AYM1.

Finally, let us note that the intervals $I_1,...,I_n$ considered in the routine AYM3 are of equal length. Thus the drawing of the number of the interval (Step 1) is equivalent to the drawing of a number x according to uniform distribution on (a,b) and checking to which interval it belongs. This remark together with the Complete Probability Formula yields the following equivalent yet very simple form of AYM3

```
Set k=1;
While k≤N Do Step 1 to Step 3
   Step 1. Set t=Random(I)
   Step 2. Set z[k]=f(t)
   Step 3. Set k=k+1
Set sample=(z[1],...,z[N])
Return sample
```

The structure of the above procedure is exactly the same as the form of the procedure DYM. The only difference now is that the function f does not have to be the simple one and thus the underlying generated measure does not have to be discrete.

In the next section we compare the three above generators as tools for simulating of the Young distribution related to piecewise functions.

4. Comparison of the Random-Number Generators

The main objective of the comparison is to identify a generator which produces distributions that best approximate the true underlying Young measures. In order to achieve our goal we have analyzed dozens of benchmarking problems, i.e. problems where the explicit form of the Young measure is known. Here we present just two examples of such ones.

As an index of quality of the generated distributions we use the Pearson goodness-of-fit chi-square statistic (as usual denoted by $\chi 2$).

Example 1.
In this benchmarking problem we use a function *f* defined on the interval I = [0,4] given by the formula:

$$f(x) = \begin{cases} 4x & , \quad x < 1/4 \\ x + 3/4 & , \quad 1/4 \le x < 5/4 \\ -2x + 9/2 & , \quad 5/4 \le x < 2 \\ -x/4 + 1 & , \quad 2 \le x \end{cases} \tag{3}$$

Such a function *f* was considered in [14]. With the help of the theoretical results from Section 1, one can find the formula defining Young measure *v* related to this function. It is concentrated on the interval [0,2], and its probability density function *g* is given by:

$$g(y) = \begin{cases} 17/16 & , \quad 0 \le y < 1/2 \\ 3/16 & , \quad 1/2 \le y < 1 \\ 3/8 & , \quad 1 \le y \le 2 \end{cases} \tag{4}$$

The graph of the above function *g* is shown in Fig.1.

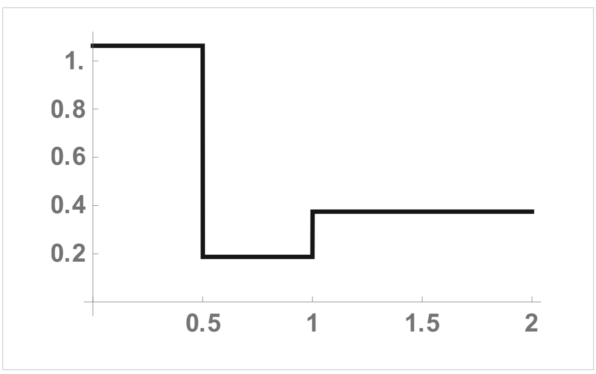

Fig. 1. Graph of the probability density function *g* given by (4)

In our comparison studies we perform simulation experiments assuming different values of the parameters *N, n* that are necessary to call the routines. In order to make the results more trustful, while applying the Pearson test we split the domain (i.e. the interval [0,2]) into 200 equal classes. Thus the sample size has to be large enough to ensure sufficient number of observation in each of the classes. Some exemplary results are presented in Table 1.

Now, let as assume the significance level of the goodness-of-fit test as $\alpha = 0.1$. Such a value of α - bigger than usually assumed - results in relatively smaller values of the probability of the type II errors, and these errors are of our primal interest. The critical value of the test in this situation is 224.957. So, if the values of the statistic $\chi 2$ are less than that, the empirical distribution generated in the experiment can be considered as similar or even identical with the theoretical one given by (4). On the other hand the smaller the values of $\chi 2$, the greater is the similarity. Apart from the $\chi 2$ criterion in Table 1 we also present the value of mean relative error, MRE, i.e. the distance between the theoretical frequencies computed on the basis of the probability density function *g* and the empirical frequencies received with the help of examined generator. However it should be emphasized that from the statistical theory point of view the $\chi 2$ criterion is of our primal interest, because it not only indicates the best generator but also tells whether the generated sample can be considered as drawn from the hypothetical theoretical distribution.

Routine	**Number of Intervals *n***	**Sample size *N***	**Statistic $\chi 2$**	**Mean Relative Error**
AYM1	250	2000	1038	3.3%
AYM2	250	2000	1028	3.5%
AYM3	250	2000	**194**	**3.0%**
AYM1	1000	2000	219	3.1%
AYM2	1000	2000	331	3.5%
AYM3	1000	2000	**172**	**3.0%**
AYM1	2500	5000	192	1.9%
AYM2	2500	5000	226	2.2%
AYM3	2500	5000	**188**	**1.9%**

Table 1. Comparison of AYM1,AYM2, and AYM3- results received for the functions described in Example 1.

When analyzing Table 1, one can see that the generator AYM3 manifests the greatest correspondence with the true Young measure considered in our benchmark. Its dominance is confirmed not only by the smallest values of the $\chi 2$ –statistic, but also by the fact that it is the only generator which produces a sample which passes the Pearson test in each presented case – the corresponding value of $\chi 2$ is less than the critical value. The values of mean relative errors are also the smallest in case of

AYM3. Figure 2 presents a histogram of a data generated with the help of AYM3 routine. Comparing it with the graph of the true theoretical distribution presented in Fig.1, we can see how accurate is the shape of the simulated distribution.

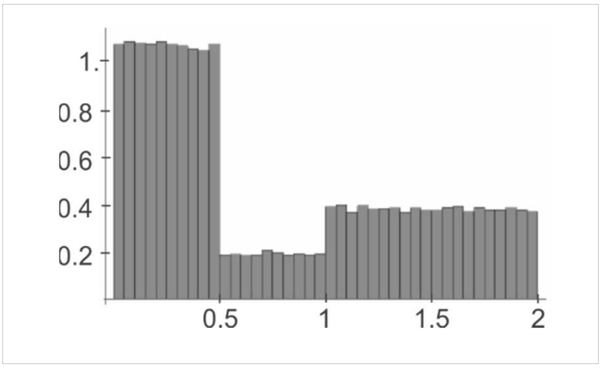

Fig. 2. The histogram based on a sample generated with the help of AYM3 with N=100000.

Example 2.
In this benchmarking problem we use a function *f* defined on the interval I = [0,6] given by the formula:

$$f(x)=\begin{cases} 8x & \text{if} \quad x<1/4 \\ 4x+1 & \text{if} \quad x\in[1/4,5/4) \\ -x+29/4 & \text{if} \quad x\in[5/4,3) \\ -13x/4+14 & \text{if} \quad x\in[3,4) \\ -x/2+3 & \text{if} \quad x\in[4,6] \end{cases} \qquad (5)$$

Again, by Theorem 1, we can find the explicit formula for the following probability density function *g* of the related Young measure. If is given by:

$$g(y)=\begin{cases} 17/48 & \text{if} \quad y\in[0,1) \\ 15/208 & \text{if} \quad y\in[1,2) \\ 29/312 & \text{if} \quad y\in[2,17/4) \\ 5/24 & \text{if} \quad y\in[17/4,6] \end{cases} \qquad (6)$$

To save the article space this time we omit the presentation of the graph of the above function *g* as well as the histograms of generated samples. We only present Table 2, which, similarly as Table 1, shows the values of the index $\chi 2$ as well as the mean relative errors. As in Example 1 we assume the significance level of the goodness-of-fit test as $\alpha= 0.1$. The critical value of the test is the same as previously: 224.957. The results presented in Table 2 confirm the dominance of the generator

AYM3. It again manifests the greatest correspondence with the true Young measure considered in our benchmark.

Routine	Number of Intervals *n*	Sample size *N*	Statistic χ2	Mean Relative Error
AYM1	250	2000	919.8	3.65%
AYM2	250	2000	1118.4	4.10%
AYM3	250	2000	**206.9**	**3.63%**
AYM1	1000	2000	269.9	3.75%
AYM2	1000	2000	302.6	3.65%
AYM3	1000	2000	**186.2**	**3.55%**
AYM1	2500	5000	217.6	2.32%
AYM2	2500	5000	201.2	2.35%
AYM3	2500	5000	**173.9**	**2.29%**

Table 2. Comparison of AYM1,AYM2, and AYM3- results received for the functions described in Example 2.

Routine	Number of Intervals *n*	Sample size *N*	Statistic χ2	Mean Relative Error
AYM1	250	2000	897.0	8.3%
AYM2	250	2000	1015.8	8.2%
AYM3	250	2000	**198.5**	**6.7%**
AYM1	1000	2000	268.1	7.3%
AYM2	1000	2000	292.0	7.2%
AYM3	1000	2000	**198.9**	**7.2%**
AYM1	2500	5000	236.8	4.4%
AYM2	2500	5000	252.7	4.4%
AYM3	2500	5000	**199.9**	**4.4%**

Table 3. Comparison of AYM1,AYM2, and AYM3- average results received for 50 randomly chosen piece-affine functions f. The number of "pieces" in each random function f was separately randomly chosen within the range 4 to 10.

The dominance of the generator AYM3 is also confirmed by other benchmarking problems considered in our studies. We have considered dozens of such, and in each case the generator AYM3 proved to be the most accurate one. The summary (average) results obtained for 50 randomly chosen piece-affine functions f are presented in

Table 3. The number of "pieces" in these random functions *f* was also randomly chosen within the range 4 to 10.

5. Final Remarks

The performance of AYM1 and AYM2 looks rather poor. We see that AYM1 passes the Pearson test in only two cases, while the AYM2 only once. However, the quality of the samples generated by AYM1 and AYM2 improves when the number of intervals (along with sample size) increases - the AYM1 performs really well for $n \geq 10000$ and N~500000, see [14]. Nonetheless, in each situation it performs worse than AYM3.

The AYM3 can be used not only to produce histograms, but obviously it can be also helpful in estimating various characteristics of the underlying Young measure (from our experience: the relative error in estimating the expected values and standard deviations was less than 1% in all our benchmarking problems). What is even more important, generating the Young measure enables us to receive very accurate approximations of the values of the integrals given in (1) – a primal and practically the most important purpose for which we need the Young measure. It is well-known that the integral $\int_K \beta(k)d\nu(k)$, with ν being the homogenous Young measure, can be very efficiently approximated by the average of the sample of pseudo-random numbers β(K), where K is generated according to the Young measure ν. Consequently, the construction of good Young measure generator is an important task. The AYM3 manifests really very good performance, which, especially in relation to AYM2 (which at first look, seems to be very similar), may be a bit surprising. However a more profound analysis of the differences between these two algorithms leads us to very interesting and important theoretical conclusions. Our research hypothesis is the following: a Young measure related to any Borel function *f* defined on the interval *I* is the distribution of the random variable *f(U)*, where U has got the uniform distribution on *I*. If the statement is true then no other generator could better simulate the Young measures than the AYM3. We hope to present the proof of our hypothesis very soon. However it requires usage of many notions and tools from the functional analysis. So when it is completed, it will be published in separate paper.

References

[1] L. C. Young, "Generalized curves and the existence of an attained absolute minimum in the calculus of variations", C. R. Soc. Sci. Lett. Varsovie, Classe III 30, pp 212-234, 1937.

[2] C. Castaing, P. Raynaud de Fitte, M. Valadier, Young Measures on Topological Spaces, Kluwer Academic Publishers, Dordrecht, Boston, London 2004.

[3] L. C. Florescu, C. Godet-Thobie, Young measures and Compactness in Measure Spaces, Walter de Gruyter GmbH & Co. KG, Berlin, Boston 2012.

[4] P. Pedregal, Parametrized Measures and Variational Principles, Progress in Nonlinear Differential Equations and Their Applications, 30, Birkhäuser Verlag, Basel 1997.

[5] T. Roubíček, Relaxation in Optimization Theory and Variational Calculus, Walter de Gruyter, Berlin, New York, 1997.

[6] H. Attouch, G. Buttazzo, G. Michaille, Variational Analysis in Sobolev and BV Spaces:Applications to PDEs and Optimization, SIAM and MPS, Philadelphia, USA 2006.

[7] J. Málek, J. Nečas, M. Rokyta, M. Růžička, Weak and Measure-valued Solutions to Evolutionary PDEs, Chapman & Hall 1996.

[8] S. Müller, "Variational methods for microstructure and phase transitions", in Calculus of Variations and Geometric Evolution Problems, Lecture Notes in Mathematics vol. 1713, S. Hildebrandt, M. Struwe, Eds. Springer Verlag, Berlin, Heidelberg, Germany, 1999, pp.85-210.

[9] P. Pedregal, Variational Methods in Nonlinear Elasticity, SIAM, Philadelphia, USA, 2000.

[10] S. Bartels, Numerical Methods for Nonlinear Partial Differential Equations, Springer Series on Computational Mathematics, vol. 47, Springer International Publishing Switzerland 2015.

[11] C. Carstensen, "Numerical analysis of microstructure", in Theory and Numerics of Differential Equations, Durham 2000, Universitext, J. F. Blowey, J. P. Coleman, A. W. Craig, Eds. Springer Verlag, Berlin, Germany, 2000, pp. 60-126.

[12] E. J. Balder, "Lecture on Young measures theory and its application in economics", Rend. Instit. Mat. Unov. Trieste, vol. 31 suppl. 1, pp. 1-69, 2000.

[13] P. Puchała, "An elementary method of calculating an explicit form of Young measures in some special cases", Optimization, vol. 63, no. 9, pp. 1419-1430, 2014.

[14] A. Grzybowski, P. Puchała, "Remarks about discrete Young measures and their Monte Carlo simulations", Journal of Applied Mathematics and Computational Mechanics, vol. 14, no. 2, pp. 13-20, 2015. Available: http://amcm.pcz.pl/?id=view&volume=14&issue=2&article=2