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Approximation of Unit-Hypercubic Infinite Antagonistic Game via Dimension-Dependent Irregular Samplings and Reshaping the Payoffs into Flat Matrix Wherewith to Solve the Matrix Game

Vadim Romanuke *romanukevadimv@mail.ru*
Applied Mathematics and Social Informatics Department
Khmelnitskiy National University, Khmelnitskiy, Ukraine

Abstract

There is suggested a method of approximating unit-hypercubic infinite antagonistic game with the matrix game and the corresponding solution in finite strategies. The method is based on dimension-dependent irregular samplings over the game kernel. Numbers of the sampling points and their coordinates are chosen due to the stated conditions. If the sampled kernel is not the flat matrix then the multidimensional payoff matrix is reshaped into the flat one wherewith to solve the matrix game. The reshaping is a reversible matrix map. The concluding step of the approximation is in checking the finite solution consistency. Consistency can be weakened, and consistency can be checked at some rank corresponding to a natural number. The higher rank of the finite solution consistency is, the wider neighborhood of the sampling numbers at which the solution properties are monotonic-like.
Keywords: infinite antagonistic game, unit hypercube, sampling, multidimensional matrix, matrix game, finite support strategy, approximate solution consistency

1. Infinite Antagonistic Game Models

Infinite antagonistic game model a lot of conflict events associated with resources allocation [2, 7] and military processes [10], where components of the player's action space constitute intervals. In technical-technological uncertain problems [5, 12], infinite antagonistic game is an appropriate model for removing multivariate or interval uncertainties [11, 8]. There are also time-selection games [11, 6], modeling economic-ecological interaction and sports. Sometimes, if there is a large number of pure strategies for a player, it is methodologically natural as well as practically useful to assume that the set of pure strategies for this player is infinite [7, 12, 11]. Thus game infiniteness allows modeling a wider class of conflict events, providing unrestricted possibilities to watch the event and to get its result payoff.

2. Solving Infinite Antagonistic Games on Compact Action Spaces

The general theory of infinite antagonistic games is much more complex than the theory of finite antagonistic games. However, if the antagonistic game kernel, defined on a Euclidean finite-dimensional closed subspace, is a continuous function, then the game has a solution at least in mixed strategies [11]. And convex-concave games on such subspaces are solved easily: one of the players has the pure optimal strategy, and another player has the mixed strategy — an optimal mixture of a few pure strategies [10, 12]. Generally, any continuous compact game can be solved in ε-optimal strategies with finite supports [11]. And every completely bounded antagonistic game has ε-optimal strategies with finite supports [7, 12, 11]. Support finiteness is a very important property, letting implement the solution statistically if the game is played repeatedly. If support is infinite, there is statistical impossibility to use every pure strategy from the support at least singly, what shifts the averaged game value against the optimal one.

Nonetheless, compact games cannot be solved by a universal algorithmic approach, unless they are finite games [3]. Construction of ε-nets [3] over compact brings a game finite approximation, whose properties depend on value of the taken ε. However, it is unclear how to select ε for the construction that the ε-solution be sufficiently close to the genuine solution of the initial infinite game. Sufficiency of the closeness is uncertain as well.

3. Goal of Article and Tasks for Its Accomplishment

This paper goal is to state conditions for finite approximation of infinite compact antagonistic game, using similar to ε-net-construction technique, but without resorting to Helly metric for promptitude. A new approximation technique should be a method of sampling the infinite antagonistic game, where the sampling measure is chosen explicitly. For compact game case, it is easy to find an isomorphic game to the initial one, that the set of every player's pure strategies would be Euclidean finite-dimensional closed and bounded subspace [12, 11, 1]. Normally, that the spoken subspace is unit cube of the appropriate dimension [12, 13].

Approximation of unit-hypercubic infinite antagonistic game is going to be accomplished via execution of the following tasks:

1. The game kernel, defined on unit hypercube, must be sampled. In general, there is no constant sampling step along a hypercube dimension. The number of points to be sampled will be specific for each dimension. Therefore, conditions for dimension-dependent irregular samplings will be stated, letting sample the kernel correctly and accurately.
2. The multidimensional matrix of payoff values must be reshaped into flat matrix wherewith to solve the matrix game. Maintenance of one-to-one indexing should be provided.
3. The solution of the matrix game, being the approximation of unit-hypercubic infinite antagonistic game, mustn't be too dependent upon the sampling. Thus, the players' optimal strategies will be checked for their consistency, implying relative independence upon the sampling within minimal neighborhoods of numbers of points to be sampled.
4. The consistency conception should be generalized off the minimal neighborhoods to wider ranges of natural numbers.

Finally, discussible and completive notices are going to be declared. These declarations should clarify the paper scientific contribution and outline ways for further research.

4. Sampling the Infinite Antagonistic Game Irregularly

Let

$$H_M^{\langle 1 \rangle} = \bigtimes_{m=1}^{M} [0;\, 1] \subset \mathbb{R}^M \tag{1}$$

and

$$H_N^{\langle 2 \rangle} = \bigtimes_{n=1}^{N} [0;\, 1] \subset \mathbb{R}^N \tag{2}$$

be the first and second players' pure strategy sets, respectively, where $M \in \mathbb{N}$, $N \in \mathbb{N}$. Denote the first player pure strategy by

$$\mathbf{X} = \left[x_m \right]_{1\times M} \in H_M^{\langle 1 \rangle} \tag{3}$$

and the second player pure strategy by

$$\mathbf{Y} = \left[y_n \right]_{1\times N} \in H_N^{\langle 2 \rangle}\,. \tag{4}$$

Classically, a strategy here is a real point in the player's hypercube associated with a decision or a behavioral operation (action) of the player [10, 12, 11, 7]. In the situation

$$\{\mathbf{X},\, \mathbf{Y}\} \in H_M^{\langle 1 \rangle} \times H_N^{\langle 2 \rangle} = \mathop{\times}_{d=1}^{M+N} [0;\, 1] \subset \mathbb{R}^{M+N} \tag{5}$$

the first player gets the payoff $K(\mathbf{X},\, \mathbf{Y})$, while the second player losses the same value. The subsequent unit-hypercubic infinite antagonistic game

$$\left\langle H_M^{\langle 1 \rangle},\, H_N^{\langle 2 \rangle},\, K(\mathbf{X},\, \mathbf{Y}) \right\rangle \tag{6}$$

is assumed such that its kernel $K(\mathbf{X},\, \mathbf{Y})$ defined on the unit $(M+N)$-dimensional hypercube

$$H_{M+N} = H_M^{\langle 1 \rangle} \times H_N^{\langle 2 \rangle} = \mathop{\times}_{d=1}^{M+N} [0;\, 1] \subset \mathbb{R}^{M+N} \tag{7}$$

is measurable and is differentiable with respect to any of variables

$$\left\{ \{x_m\}_{m=1}^{M},\, \{y_n\}_{n=1}^{N} \right\}. \tag{8}$$

Also there exist mixed derivatives of function $K(\mathbf{X},\, \mathbf{Y})$ by any combination of variables (8) in any situation (5), where every variable is included no more than just once.

While sampling irregularly, let $S_m^{\langle 1 \rangle}$ be the number of intervals between the selected points in m-th dimension of hypercube (1), and $S_n^{\langle 2 \rangle}$ be the number of intervals between the selected points in n-th dimension of hypercube (2). Remember that each dimension is the unit segment now. In the utmost case of sampling, $S_m^{\langle 1 \rangle} \in \mathbb{N}$ and $S_n^{\langle 2 \rangle} \in \mathbb{N}$. Therefore, endpoints of the unit segment are included into the sampling necessarily. There is no fixed sampling step. Thus, in m-th dimension the first player instead of the segment $[0;\, 1]$ of values of m-th component of its pure strategy (3) now possesses the set of points

$$D_m^{\langle 1 \rangle}\left(S_m^{\langle 1 \rangle}\right) = \left\{ x_m^{\langle s_m \rangle} \right\}_{s_m=1}^{S_m^{\langle 1 \rangle}+1} \text{ by } x_m^{\langle 1 \rangle} = 0,\; x_m^{\left\langle S_m^{\langle 1 \rangle}+1 \right\rangle} = 1,$$
$$x_m^{\langle d_m \rangle} < x_m^{\langle d_m + 1 \rangle} \quad \forall\, d_m = \overline{1,\, S_m^{\langle 1 \rangle}} \text{ at } m = \overline{1,\, M}. \tag{9}$$

In n-th dimension the second player instead of the segment $[0;\, 1]$ of values of n-th component of its pure strategy (4) now possesses the set of points

$$D_n^{\langle 2 \rangle}\left(S_n^{\langle 2 \rangle}\right) = \left\{ y_n^{\langle s_n \rangle} \right\}_{s_n=1}^{S_n^{\langle 2 \rangle}+1} \text{ by } y_n^{\langle 1 \rangle} = 0,\; y_n^{\left\langle S_n^{\langle 2 \rangle}+1 \right\rangle} = 1,$$
$$y_n^{\langle d_n \rangle} < y_n^{\langle d_n + 1 \rangle} \quad \forall\, d_n = \overline{1,\, S_n^{\langle 2 \rangle}} \text{ at } n = \overline{1,\, N}. \tag{10}$$

Consequently, the finite hypercubic irregular lattice

$$D^{\langle 1 \rangle} = \mathop{\times}_{m=1}^{M} D_m^{\langle 1 \rangle}\left(S_m^{\langle 1 \rangle}\right) = \mathop{\times}_{m=1}^{M} \left\{ \left\{ x_m^{\langle s_m \rangle} \right\}_{s_m=1}^{S_m^{\langle 1 \rangle}+1} \right\} \subset H_M^{\langle 1 \rangle} \tag{11}$$

substitutes the hypercube (1), and the finite hypercubic irregular lattice

$$D^{\langle 2 \rangle} = \mathop{\times}_{n=1}^{N} D_n^{\langle 2 \rangle}\left(S_n^{\langle 2 \rangle}\right) = \mathop{\times}_{n=1}^{N} \left\{ \left\{ y_n^{\langle s_n \rangle} \right\}_{s_n=1}^{S_n^{\langle 2 \rangle}+1} \right\} \subset H_N^{\langle 2 \rangle} \tag{12}$$

substitutes the hypercube (2). This makes possible transition from infinite game (6) to finite one

$$\left\langle D^{\langle 1 \rangle}, D^{\langle 2 \rangle}, K(\mathbf{X}, \mathbf{Y}) \right\rangle \text{ by } \mathbf{X} \in D^{\langle 1 \rangle} \text{ and } \mathbf{Y} \in D^{\langle 2 \rangle} \tag{13}$$

on the finite hypercubic irregular lattice $D^{\langle 1 \rangle} \times D^{\langle 2 \rangle} \subset H_{M+N}$.

The choice of numbers

$$\left\{ \left\{ S_m^{\langle 1 \rangle} \right\}_{m=1}^{M}, \left\{ S_n^{\langle 2 \rangle} \right\}_{n=1}^{N} \right\} \tag{14}$$

shouldn't erase information about local extremums and gradient over hypersurface $K(\mathbf{X}, \mathbf{Y})$. That is $\forall\, s_m = \overline{1, S_m^{\langle 1 \rangle}}$ and $\forall\, s_n = \overline{1, S_n^{\langle 2 \rangle}}$ there ought to be

$$\frac{\partial^{M+N} K(\mathbf{X}, \mathbf{Y})}{\partial x_1 \partial x_2 \dots \partial x_M \partial y_1 \partial y_2 \dots \partial y_N} \geqslant 0 \text{ or } \frac{\partial^{M+N} K(\mathbf{X}, \mathbf{Y})}{\partial x_1 \partial x_2 \dots \partial x_M \partial y_1 \partial y_2 \dots \partial y_N} \leqslant 0$$
$$\forall\, x_m \in \left[x_m^{\langle s_m \rangle}; x_m^{\langle s_m + 1 \rangle} \right] \text{ and } \forall\, y_n \in \left[y_n^{\langle s_n \rangle}; y_n^{\langle s_n + 1 \rangle} \right] \text{ by } m = \overline{1, M} \text{ and } n = \overline{1, N}, \tag{15}$$

and

$$\left| \frac{\partial^{M+N} K(\mathbf{X}, \mathbf{Y})}{\partial x_1 \partial x_2 \dots \partial x_M \partial y_1 \partial y_2 \dots \partial y_N} \right| \leqslant \gamma$$
$$\forall\, x_m \in \left[x_m^{\langle s_m \rangle}; x_m^{\langle s_m + 1 \rangle} \right] \text{ and } \forall\, y_n \in \left[y_n^{\langle s_n \rangle}; y_n^{\langle s_n + 1 \rangle} \right] \text{ by } m = \overline{1, M} \text{ and } n = \overline{1, N} \tag{16}$$

for some $\gamma > 0$, implying tolerable fluctuations of the hypersurface $K(\mathbf{X}, \mathbf{Y})$ on every one of segments

$$\left\{ \left\{ \left\{ \left[x_m^{\langle s_m \rangle}; x_m^{\langle s_m + 1 \rangle} \right] \right\}_{s_m = 1}^{S_m^{\langle 1 \rangle}} \right\}_{m=1}^{M}, \left\{ \left\{ \left[y_n^{\langle s_n \rangle}; y_n^{\langle s_n + 1 \rangle} \right] \right\}_{s_n = 1}^{S_n^{\langle 2 \rangle}} \right\}_{n=1}^{N} \right\}.$$

Actually, inequalities (15) and (16) are conditions for sampling the game (6) kernel from the hypercube (7) down to hypercubic irregular lattice $D^{\langle 1 \rangle} \times D^{\langle 2 \rangle}$ with (9) — (12). The following assertion directs for choosing the numbers (14) and points

$$\left\{ \left\{ \left\{ x_m^{\langle s_m \rangle} \right\}_{s_m = 1}^{S_m^{\langle 1 \rangle} + 1} \right\}_{m=1}^{M}, \left\{ \left\{ y_n^{\langle s_n \rangle} \right\}_{s_n = 1}^{S_n^{\langle 2 \rangle} + 1} \right\}_{n=1}^{N} \right\} \tag{17}$$

in order to sample the game (6) kernel.

Theorem 1. If the hypersurface $K(\mathbf{X}, \mathbf{Y})$ local extremums are reached at points having only components

$$\left\{ \left\{ \left\{ x_m^{\langle s_m \rangle} \right\}_{s_m = 2}^{S_m^{\langle 1 \rangle}} \right\}_{m=1}^{M}, \left\{ \left\{ y_n^{\langle s_n \rangle} \right\}_{s_n = 2}^{S_n^{\langle 2 \rangle}} \right\}_{n=1}^{N} \right\} \tag{18}$$

then, if inequalities (16) hold, the game (6) kernel is sampled with (9) — (12).

Proof. Since having local extremums only with components (18), the hypersurface $K(\mathbf{X}, \mathbf{Y})$ does not have local extremums on every one of intervals

$$\left\{ \left\{ \left\{ \left(x_m^{\langle s_m \rangle}; x_m^{\langle s_m + 1 \rangle} \right) \right\}_{s_m = 1}^{S_m^{\langle 1 \rangle}} \right\}_{m=1}^{M}, \left\{ \left\{ \left(y_n^{\langle s_n \rangle}; y_n^{\langle s_n + 1 \rangle} \right) \right\}_{s_n = 1}^{S_n^{\langle 2 \rangle}} \right\}_{n=1}^{N} \right\}.$$

Hence, for the differentiable hypersurface $K(\mathbf{X}, \mathbf{Y})$, conditions (15) hold as well. The theorem has been proved.

For conditions (16), parameter γ is pre-assigned on reasoning about the value

$$\gamma \leqslant \lambda \cdot \left(\max_{\mathbf{X} \in H_M^{\langle 1 \rangle}} \max_{\mathbf{Y} \in H_N^{\langle 2 \rangle}} K(\mathbf{X}, \mathbf{Y}) - \min_{\mathbf{X} \in H_M^{\langle 1 \rangle}} \min_{\mathbf{Y} \in H_N^{\langle 2 \rangle}} K(\mathbf{X}, \mathbf{Y}) \right)$$

by, say,

$$\lambda \in \{0.001, 0.005, 0.01, 0.02\}$$

or other values for scale factor λ. Nevertheless, some conditions below may cause need to resample the game (6) kernel. These conditions are going to be applied for the finite game (13) solution. But before stating them, this game should be represented as matrix one.

5. Mapping Payoff Multidimensional Matrix into Flat Matrix

The triplet (13) is a finite game but not matrix game unless $M = N = 1$. The multidimensional matrix of payoff values, generated by the values of hypersurface $K(\mathbf{X}, \mathbf{Y})$ on the finite lattice $D^{\langle 1 \rangle} \times D^{\langle 2 \rangle}$, must be reshaped into flat matrix. This will allow to find the finite game solution with any acceptable methods for solving matrix games, including linear programming method [7, 12, 11].

Theorem 2. If there is $(M + N)$-dimensional matrix $\mathbf{P}(0) = [p_J]_{\mathscr{F}}$ of the format

$$\mathscr{F} = \left\{ \bigtimes_{m=1}^{M} \left(S_m^{\langle 1 \rangle} + 1 \right) \right\} \times \left\{ \bigtimes_{n=1}^{N} \left(S_n^{\langle 2 \rangle} + 1 \right) \right\}, \tag{19}$$

whose $(M + N)$-position indices

$$J = \{ j_k \}_{k=1}^{M+N} \text{ by } j_m \in \left\{ \overline{1, S_m^{\langle 1 \rangle} + 1} \right\} \text{ and } j_{M+n} \in \left\{ \overline{1, S_n^{\langle 2 \rangle} + 1} \right\} \tag{20}$$

determine the matrix element

$$p_J = K(\mathbf{X}, \mathbf{Y}) \text{ by } x_m = x_m^{\langle j_m \rangle}, \; y_n = y_n^{\langle j_{M+n} \rangle} \quad \forall\, m = \overline{1, M} \text{ and } \forall\, n = \overline{1, N}, \tag{21}$$

then $\mathscr{F}$-matrix $\mathbf{P}(0) = [p_J]_{\mathscr{F}}$ is reshaped into two-dimensional matrix $\mathbf{G}(0) = \left[g_{u_1 u_2} \right]_{\mathscr{L}}$ of elements $g_{u_1 u_2} = p_J$, whose format is

$$\mathscr{L} = \left\{ \prod_{m=1}^{M} \left(S_m^{\langle 1 \rangle} + 1 \right) \right\} \times \left\{ \prod_{n=1}^{N} \left(S_n^{\langle 2 \rangle} + 1 \right) \right\}. \tag{22}$$

The matrix map $\mathbf{P}(0) \to \mathbf{G}(0)$, realizing this reshaping, is reversible.

Proof. Let the subset of indices $\{ j_m \}_{m=1}^{M} \subset J$ be convolved into value

$$u_1 = \sum_{m=1}^{M} \left(\prod_{m_1=1}^{m-1} \left(S_{M-m_1+1}^{\langle 1 \rangle} + 1 \right) \right) \cdot \left(j_{M-m+1} - \operatorname{sign}(m-1) \right) \text{ by } m = \overline{1, M}. \tag{23}$$

It is obvious that the value (23) is always integer. Moreover, for $j_m = \overline{1, S_m^{\langle 1 \rangle} + 1}$ there is

$$u_1 = \overline{1, Q_1(0)} \text{ by } Q_1(0) = \prod_{m=1}^{M} \left(S_m^{\langle 1 \rangle} + 1 \right). \tag{24}$$

Similarly, letting the subset of indices $\{ j_{M+n} \}_{n=1}^{N} \subset J$ be convolved into value

$$u_2 = \sum_{n=1}^{N} \left(\prod_{n_1=1}^{n-1} \left(S_{N-n_1+1}^{\langle 2 \rangle} + 1 \right) \right) \cdot \left(j_{M+N-n+1} - \mathrm{sign}(n-1) \right) \text{ by } n = \overline{1, N}, \tag{25}$$

the value (25) appears integer and for $j_{M+n} = \overline{1, S_n^{\langle 2 \rangle} + 1}$ there is

$$u_2 = \overline{1, Q_2(0)} \text{ by } Q_2(0) = \prod_{n=1}^{N} \left(S_n^{\langle 2 \rangle} + 1 \right). \tag{26}$$

Thus $(M+N)$-dimensional matrix $\mathbf{P}(0) = [p_J]_{\mathscr{F}}$ of the format (19) by (20) and (21) is reshaped into two-dimensional matrix $\mathbf{G}(0) = \left[g_{u_1 u_2} \right]_{\mathscr{L}}$ of the format (22), where the set of indices $J = \{j_k\}_{k=1}^{M+N}$ is mapped into the set $\{u_1, u_2\}$ by (23) and (25). Further, let the function $\rho(a, b)$ by $b \neq 0$ round the fraction $\frac{a}{b}$ to the nearest integer towards zero. And let

$$\eta(a, b) = a - b \cdot \rho(a, b).$$

Then the subset of indices $\{j_m\}_{m=1}^{M} \subset J$ is restored by the index u_1:

$$j_M = \eta\left(u_1, S_M^{\langle 1 \rangle} + 1\right) + \left(S_M^{\langle 1 \rangle} + 1\right)\left(1 - \mathrm{sign}\left[\eta\left(u_1, S_M^{\langle 1 \rangle} + 1\right)\right]\right), \tag{27}$$

$$j_{M-m} = 1 + \eta\left(\frac{u_1 - j_M - \sum_{m_1=1}^{m-1} \left(\prod_{m_2=1}^{m_1} \left(S_{M-m_2+1}^{\langle 1 \rangle} + 1 \right) \right) \cdot \left(j_{M-m_1} - 1 \right)}{\prod_{m_1=1}^{m} \left(S_{M-m_1+1}^{\langle 1 \rangle} + 1 \right)}, S_{M-m}^{\langle 1 \rangle} + 1 \right)$$

$$\forall\, m = \overline{1, M-1}. \tag{28}$$

The subset of indices $\{j_{M+n}\}_{n=1}^{N} \subset J$ is restored by the index u_2 similarly:

$$j_{M+N} = \eta\left(u_2, S_N^{\langle 2 \rangle} + 1\right) + \left(S_N^{\langle 2 \rangle} + 1\right)\left(1 - \mathrm{sign}\left[\eta\left(u_2, S_N^{\langle 2 \rangle} + 1\right)\right]\right), \tag{29}$$

$$j_{M+N-n} = 1 + \eta\left(\frac{u_2 - j_{M+N} - \sum_{n_1=1}^{n-1} \left(\prod_{n_2=1}^{n_1} \left(S_{N-n_2+1}^{\langle 2 \rangle} + 1 \right) \right) \cdot \left(j_{M+N-n_1} - 1 \right)}{\prod_{n_1=1}^{n} \left(S_{N-n_1+1}^{\langle 2 \rangle} + 1 \right)}, S_{N-n}^{\langle 2 \rangle} + 1 \right)$$

$$\forall\, n = \overline{1, N-1}. \tag{30}$$

Consequently, the matrix map $\mathbf{P}(0) \to \mathbf{G}(0)$ is accomplished via (23) and (25), and the matrix map $\mathbf{G}(0) \to \mathbf{P}(0)$ is accomplished via (27) — (30). The theorem has been proved.

When $M = N = 1$, Theorem 2 is useless — the finite game (13) is already flat matrix game. When $M \neq 1$ or $N \neq 1$, Theorem 2 allows mapping the finite game (13) on the finite hypercubic irregular lattice $D^{\langle 1 \rangle} \times D^{\langle 2 \rangle}$ into matrix game

$$\left\langle \left\{ z_{u_1}^{\langle \mathbf{X} \rangle}(0) \right\}_{u_1=1}^{Q_1(0)}, \left\{ z_{u_2}^{\langle \mathbf{Y} \rangle}(0) \right\}_{u_2=1}^{Q_2(0)}, \mathbf{G}(0) \right\rangle, \tag{31}$$

where the first player's pure strategy $z_{u_1}^{\langle \mathbf{X} \rangle}(0)$ corresponds to its strategy $\mathbf{X} = \left[x_m^{\langle j_m \rangle} \right]_{1 \times M}$ in the initial game (6) after having sampled under numbers $\left\{ S_m^{\langle 1 \rangle} \right\}_{m=1}^{M}$, and the second player's pure strategy $z_{u_2}^{\langle \mathbf{Y} \rangle}(0)$ corresponds to its strategy $\mathbf{Y} = \left[y_n^{\langle j_{M+n} \rangle} \right]_{1 \times N}$ in the initial game (6) after having sampled under numbers $\left\{ S_n^{\langle 2 \rangle} \right\}_{n=1}^{N}$. The section below aims to learn how good the game (31) is to be approximation of the infinite game (6).

6. Consistency of Finite Support Optimal Strategy, Approximating the Player's Genuine Optimal Strategy

Speaking generally, the genuine optimal value in the game (6) may be unknown or non-existent. The same concerns the players' genuine optimal strategies — they may exist only as ε-optimal ones. That's why we can't compare the solution

$$\left\{ \left\{ p_*(u_1, 0) \right\}_{u_1=1}^{Q_1(0)}, \left\{ q_*(u_2, 0) \right\}_{u_2=1}^{Q_2(0)} \right\} \tag{32}$$

of the game (31) to anything else, in which $p_*(u_1, 0)$ is the optimal probability of applying the pure strategy $z_{u_1}^{\langle \mathbf{X} \rangle}(0)$, and $q_*(u_2, 0)$ is the optimal probability of applying the pure strategy $z_{u_2}^{\langle \mathbf{Y} \rangle}(0)$, producing the game (31) optimal value

$$v_*(0) = \sum_{u_1=1}^{Q_1(0)} \sum_{u_2=1}^{Q_2(0)} g_{u_1 u_2} \cdot p_*(u_1, 0) \cdot q_*(u_2, 0) = \sum_{u_1^* \in U_1(0)} \sum_{u_2^* \in U_2(0)} g_{u_1^* u_2^*} \cdot p_*(u_1^*, 0) \cdot q_*(u_2^*, 0)$$

by denoting supports

$$\operatorname{supp} \left\{ p_*(u_1, 0) \right\}_{u_1=1}^{Q_1(0)} = \left\{ z_{u_1^*}^{\langle \mathbf{X} \rangle}(0) \right\}_{u_1^* \in U_1(0) \subset \left\{ \overline{1, Q_1(0)} \right\}}$$

and

$$\operatorname{supp} \left\{ q_*(u_2, 0) \right\}_{u_2=1}^{Q_2(0)} = \left\{ z_{u_2^*}^{\langle \mathbf{Y} \rangle}(0) \right\}_{u_2^* \in U_2(0) \subset \left\{ \overline{1, Q_2(0)} \right\}}.$$

Nonetheless, having no pivot, a comparison can be done over a few matrix games, approximating the initial one (6). For that we should consider δ-matrix game

$$\left\langle \left\{ z_{u_1}^{\langle \mathbf{X} \rangle}(\delta) \right\}_{u_1=1}^{Q_1(\delta)}, \left\{ z_{u_2}^{\langle \mathbf{Y} \rangle}(\delta) \right\}_{u_2=1}^{Q_2(\delta)}, \mathbf{G}(\delta) \right\rangle \tag{33}$$

by

$$Q_1(\delta) = \prod_{m=1}^{M} \left(S_m^{\langle 1 \rangle} + 1 + \delta \right), \quad Q_2(\delta) = \prod_{n=1}^{N} \left(S_n^{\langle 2 \rangle} + 1 + \delta \right),$$

which is built by $\delta \in \mathbb{Z} \setminus \{0\}$ and re-finding (9), (10), and re-mapping $\mathbf{P}(\delta) \to \mathbf{G}(\delta)$ with identifications

$$\left\{ S_m^{\langle 1 \rangle} \equiv S_m^{\langle 1 \rangle} + \delta \right\}_{m=1}^{M}, \quad \left\{ S_n^{\langle 2 \rangle} \equiv S_n^{\langle 2 \rangle} + \delta \right\}_{n=1}^{N}, \tag{34}$$

whereupon the first player's pure strategy $z_{u_1}^{\langle \mathbf{X} \rangle}(\delta)$ corresponds to its strategy $\mathbf{X} = \left[x_m^{\langle j_m \rangle} \right]_{1 \times M}$ in the initial game (6) after having sampled under numbers $\left\{ S_m^{\langle 1 \rangle} + \delta \right\}_{m=1}^{M}$, and the second player's pure strategy $z_{u_2}^{\langle \mathbf{Y} \rangle}(\delta)$ corresponds to its strategy $\mathbf{Y} = \left[y_n^{\langle j_{M+n} \rangle} \right]_{1 \times N}$ in the initial game (6) after having sampled under numbers $\left\{ S_n^{\langle 2 \rangle} + \delta \right\}_{n=1}^{N}$. And may there be a convention that the numbers

$$\left\{ \left\{ S_m^{\langle 1 \rangle} + \delta \right\}_{m=1}^{M}, \left\{ S_n^{\langle 2 \rangle} + \delta \right\}_{n=1}^{N} \right\} \tag{35}$$

are chosen against the numbers (14) so that density of the sampling points along each dimension by $\delta > 0$ doesn't decrease, and density of the sampling points along each dimension by $\delta < 0$ doesn't increase. That is, for points

$$\left\{ \left\{ \left\{ x_m^{\langle s_m \rangle}(\delta) \right\}_{s_m=1}^{S_m^{\langle 1 \rangle}+1} \right\}_{m=1}^{M}, \left\{ \left\{ y_n^{\langle s_n \rangle}(\delta) \right\}_{s_n=1}^{S_n^{\langle 2 \rangle}+1} \right\}_{n=1}^{N} \right\} \tag{36}$$

chosen after the numbers (35) with $\delta \in \mathbb{N}$, the inequalities

$$\max_{d_m = \overline{1,\, S_m^{\langle 1 \rangle}}} \left(x_m^{\langle d_m + 1 \rangle} - x_m^{\langle d_m \rangle} \right) \geqslant \max_{d_m = \overline{1,\, S_m^{\langle 1 \rangle} + \delta}} \left(x_m^{\langle d_m + 1 \rangle}(\delta) - x_m^{\langle d_m \rangle}(\delta) \right) \text{ at } m = \overline{1, M} \tag{37}$$

and

$$\max_{d_n = \overline{1,\, S_n^{\langle 2 \rangle}}} \left(y_n^{\langle d_n + 1 \rangle} - y_n^{\langle d_n \rangle} \right) \geqslant \max_{d_n = \overline{1,\, S_n^{\langle 2 \rangle} + \delta}} \left(y_n^{\langle d_n + 1 \rangle}(\delta) - y_n^{\langle d_n \rangle}(\delta) \right) \text{ at } n = \overline{1, N} \tag{38}$$

hold.

May the set

$$\left\{ \left\{ p_*(u_1, \delta) \right\}_{u_1=1}^{Q_1(\delta)}, \left\{ q_*(u_2, \delta) \right\}_{u_2=1}^{Q_2(\delta)} \right\} \tag{39}$$

be the game (33) solution, in which $p_*(u_1, \delta)$ is the optimal probability of applying the pure strategy $z_{u_1}^{\langle \mathbf{X} \rangle}(\delta)$, and $q_*(u_2, \delta)$ is the optimal probability of applying the pure strategy $z_{u_2}^{\langle \mathbf{Y} \rangle}(\delta)$, producing the game (33) optimal value

$$v_*(\delta) = \sum_{u_1=1}^{Q_1(\delta)} \sum_{u_2=1}^{Q_2(\delta)} g_{u_1 u_2} \cdot p_*(u_1, \delta) \cdot q_*(u_2, \delta) = \sum_{u_1^* \in U_1(\delta)} \sum_{u_2^* \in U_2(\delta)} g_{u_1^* u_2^*} \cdot p_*(u_1^*, \delta) \cdot q_*(u_2^*, \delta)$$

by denoting supports

$$\operatorname{supp} \left\{ p_*(u_1, \delta) \right\}_{u_1=1}^{Q_1(\delta)} = \left\{ z_{u_1^*}^{\langle \mathbf{X} \rangle}(\delta) \right\}_{u_1^* \in U_1(\delta) \subset \left\{ \overline{1, Q_1(\delta)} \right\}}$$

and

$$\operatorname{supp} \left\{ q_*(u_2, \delta) \right\}_{u_2=1}^{Q_2(\delta)} = \left\{ z_{u_2^*}^{\langle \mathbf{Y} \rangle}(\delta) \right\}_{u_2^* \in U_2(\delta) \subset \left\{ \overline{1, Q_2(\delta)} \right\}}.$$

For acceptance of the game (31) solution (32) as an approximate solution of the game (6), the value $v_*(0)$ and the optimal situation (32) itself mustn't vary too much as the sampling numbers (14) vary themselves. This implies relative independence of the solution (32) upon the sampling. Partially, within minimal neighborhoods of numbers of points to be sampled, it is provided by conditions

$$\left|v_*(0)-v_*(1)\right| \leqslant \left|v_*(-1)-v_*(0)\right| \tag{40}$$

and either

$$\left|U_r(1)\right| \geqslant \left|U_r(0)\right| \text{ by } r \in \{1, 2\} \tag{41}$$

or

$$\left|U_r(1)\right| \geqslant \left|U_r(0)\right| \geqslant \left|U_r(-1)\right| \text{ by } r \in \{1, 2\}. \tag{42}$$

Inequalities (41) and (42) provide logical non-decrement of the support cardinality as the sampling grows up minimally, but they don't regard the support configuration. Meanwhile, any of the supports (involving the corresponding probabilities)

$$\left\{p_*\left(u_1^*, 0\right)\right\}_{u_1^* \in U_1(0)}, \tag{43}$$

$$\left\{q_*\left(u_2^*, 0\right)\right\}_{u_2^* \in U_2(0)} \tag{44}$$

may have configuration, differing significantly from the genuine optimal support in the game (6) or from the support, obtained after different sampling. Hence, the player's supports must be relatively independent in their configuration along with (41) or (42).

The support configuration lies in representing the player's support as a hypersurface. Let for the r-th player there be a piecewise linear hypersurface $h_r(u_r, 0)$, obtained after sampling with numbers (14). Vertices of hypersurface $h_1(u_1, 0)$ are in points

$$\left\{\left\{x_m^{\langle j_m \rangle}\right\}_{m=1}^{M}, p_*(u_1, 0)\right\}_{u_1=1}^{Q_1(0)} \tag{45}$$

in the space $\mathbb{R}^{M+1}$, and vertices of hypersurface $h_2(u_2, 0)$ are in points

$$\left\{\left\{y_n^{\langle j_{M+n} \rangle}\right\}_{n=1}^{N}, q_*(u_2, 0)\right\}_{u_2=1}^{Q_2(0)} \tag{46}$$

in the space $\mathbb{R}^{N+1}$. The first player's optimal strategy support as (43) scores up the nonzero vertices of the hypersurface $h_1(u_1, 0)$ by

$$p_*(u_1, 0) = 0 \quad \forall\, u_1 \notin U_1(0),$$

matching the index $u_1^* \in U_1(0)$ to the point

$$\mathbf{X}_q(0) = \left[x_m^{\langle q \rangle}(0)\right]_{1\times M} = \left[x_m^{\langle j_m(q, 0) \rangle}\right]_{1\times M} \in H_M^{\langle 1 \rangle} \text{ by } q = \overline{1, Q_1^*(0)} \tag{47}$$

at $Q_1^*(0) = \left|U_1(0)\right|$ through expanding the index u_1^* via (27) and (28) into subset $\left\{j_m(q, 0)\right\}_{m=1}^{M} \subset J$. The second player's optimal strategy support as (44) scores up the nonzero vertices of the hypersurface $h_2(u_2, 0)$ by

$$q_*(u_2, 0) = 0 \quad \forall\, u_2 \notin U_2(0),$$

matching the index $u_2^* \in U_2(0)$ to the point

$$\mathbf{Y}_w(0) = \left[y_n^{\langle w \rangle}(0)\right]_{1\times N} = \left[y_n^{\langle j_{M+n}(w, 0) \rangle}\right]_{1\times N} \in H_N^{\langle 2 \rangle} \text{ by } w = \overline{1, Q_2^*(0)} \tag{48}$$

at $Q_2^*(0) = |U_2(0)|$ through expanding the index u_2^* via (29) and (30) into subset $\{j_{M+n}(w, 0)\}_{n=1}^{N} \subset J$. Then may the first player's set $\{\mathbf{X}_q(0)\}_{q=1}^{Q_1^*(0)}$ of the points (47) be sorted into the set

$$\{\bar{\mathbf{X}}_q(0)\}_{q=1}^{Q_1^*(0)} = \left\{\left[x_m^{\langle \bar{j}_m(q, 0)\rangle}\right]_{1\times M}\right\}_{q=1}^{Q_1^*(0)} \cap \{\mathbf{X}_q(0)\}_{q=1}^{Q_1^*(0)} = \{\mathbf{X}_q(0)\}_{q=1}^{Q_1^*(0)} \subset H_M^{\langle 1\rangle} \tag{49}$$

so that the value

$$\min_{q_1 \in \overline{\{q+1, Q_1^*(0)\}}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(0), \bar{\mathbf{X}}_{q_1}(0)\right) = \min_{q_1 \in \overline{\{q+1, Q_1^*(0)\}}} \left\|\bar{\mathbf{X}}_q(0) - \bar{\mathbf{X}}_{q_1}(0)\right\| =$$
$$= \min_{q_1 \in \overline{\{q+1, Q_1^*(0)\}}} \sqrt{\sum_{m=1}^{M}\left(x_m^{\langle \bar{j}_m(q, 0)\rangle} - x_m^{\langle \bar{j}_m(q_1, 0)\rangle}\right)^2} \tag{50}$$

with the re-sorted subset

$$\{\bar{j}_m(q, 0)\}_{m=1}^{M} \cap \{j_m(q, 0)\}_{m=1}^{M} = \{j_m(q, 0)\}_{m=1}^{M} \subset J$$

is reached at $q_1 = q+1$ for each $q = \overline{1, Q_1^*(0)-1}$ by $Q_1^*(0) < Q_1(0)$. Similarly, the second player's set $\{\mathbf{Y}_w(0)\}_{w=1}^{Q_2^*(0)}$ of the points (48) is sorted into the set

$$\{\bar{\mathbf{Y}}_w(0)\}_{w=1}^{Q_2^*(0)} = \left\{\left[y_n^{\langle \bar{j}_{M+n}(w, 0)\rangle}\right]_{1\times N}\right\}_{w=1}^{Q_2^*(0)} \cap \{\mathbf{Y}_w(0)\}_{w=1}^{Q_2^*(0)} = \{\mathbf{Y}_w(0)\}_{w=1}^{Q_2^*(0)} \subset H_N^{\langle 2\rangle} \tag{51}$$

so that the value

$$\min_{w_1 \in \overline{\{w+1, Q_2^*(0)\}}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(0), \bar{\mathbf{Y}}_{w_1}(0)\right) = \min_{w_1 \in \overline{\{w+1, Q_2^*(0)\}}} \left\|\bar{\mathbf{Y}}_w(0) - \bar{\mathbf{Y}}_{w_1}(0)\right\| =$$
$$= \min_{w_1 \in \overline{\{w+1, Q_2^*(0)\}}} \sqrt{\sum_{n=1}^{N}\left(y_n^{\langle \bar{j}_{M+n}(w, 0)\rangle} - y_n^{\langle \bar{j}_{M+n}(w_1, 0)\rangle}\right)^2} \tag{52}$$

with the re-sorted subset

$$\{\bar{j}_{M+n}(w, 0)\}_{n=1}^{N} \cap \{j_{M+n}(w, 0)\}_{n=1}^{N} = \{j_{M+n}(w, 0)\}_{n=1}^{N} \subset J \tag{53}$$

is reached at $w_1 = w+1$ for each $w = \overline{1, Q_2^*(0)-1}$ by $Q_2^*(0) < Q_2(0)$. Importantly, one ought to be aware of that the result of sorting in (49) and (51) depends on selection of the initial points $\bar{\mathbf{X}}_1(0) \in \{\mathbf{X}_q(0)\}_{q=1}^{Q_1^*(0)}$ and $\bar{\mathbf{Y}}_1(0) \in \{\mathbf{Y}_w(0)\}_{w=1}^{Q_2^*(0)}$. And it is convenient to say that

$$\bar{\mathbf{X}}_q(0) = \mathbf{X}_q(0) \quad \forall\, q = \overline{1, Q_1^*(0)} \quad \text{by} \quad Q_1^*(0) = Q_1(0) \tag{54}$$

and

$$\bar{\mathbf{Y}}_w(0) = \mathbf{Y}_w(0) \quad \forall\, w = \overline{1, Q_2^*(0)} \quad \text{by} \quad Q_2^*(0) = Q_2(0). \tag{55}$$

Considering δ-matrix games (33), let the hypersurfaces $\{h_r(u_r, \delta)\}_{r\in\{1, 2\}}$ and sets $\{\bar{\mathbf{X}}_q(\delta)\}_{q=1}^{Q_1^*(\delta)}$ and $\{\bar{\mathbf{Y}}_w(\delta)\}_{w=1}^{Q_2^*(\delta)}$ regard built and found with identifications (34) and turning to description for (45) — (53). And then there is a way to learn how good the game (31) is to be the infinite game (6) approximation, invoking minimal number of δ-matrix games, approximating the initial one (6).

Definition 1. The solution (32) of the game (31) is called weakly consistent for being the approximate solution of the game (6) if the inequalities

$$\max_{q\in\left\{\overline{1,\, Q_1^*(1)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(1),\, \overline{\mathbf{X}}_{q+1}(1)\right) \leqslant \max_{q\in\left\{\overline{1,\, Q_1^*(0)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(0),\, \overline{\mathbf{X}}_{q+1}(0)\right), \tag{56}$$

$$\max_{w\in\left\{\overline{1,\, Q_2^*(1)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(1),\, \overline{\mathbf{Y}}_{w+1}(1)\right) \leqslant \max_{w\in\left\{\overline{1,\, Q_2^*(0)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(0),\, \overline{\mathbf{Y}}_{w+1}(0)\right), \tag{57}$$

$$\max_{H_M^{\langle 1\rangle}}\left|h_1(u_1,\, 0)-h_1(u_1,\, 1)\right| \leqslant \max_{H_M^{\langle 1\rangle}}\left|h_1(u_1,\, -1)-h_1(u_1,\, 0)\right|, \tag{58}$$

$$\max_{H_N^{\langle 2\rangle}}\left|h_2(u_2,\, 0)-h_2(u_2,\, 1)\right| \leqslant \max_{H_N^{\langle 2\rangle}}\left|h_2(u_2,\, -1)-h_2(u_2,\, 0)\right|, \tag{59}$$

and

$$\left\|h_1(u_1,\, 0)-h_1(u_1,\, 1)\right\| \leqslant \left\|h_1(u_1,\, -1)-h_1(u_1,\, 0)\right\| \text{ in } \mathbb{L}_2\left(H_M^{\langle 1\rangle}\right), \tag{60}$$

$$\left\|h_2(u_2,\, 0)-h_2(u_2,\, 1)\right\| \leqslant \left\|h_2(u_2,\, -1)-h_2(u_2,\, 0)\right\| \text{ in } \mathbb{L}_2\left(H_N^{\langle 2\rangle}\right) \tag{61}$$

are true along with (40) and (41). The game (31) solution (32) is called weakly 1-consistent. Every strategy and its support in the weakly 1-consistent solution are called weakly consistent or weakly 1-consistent.

Noting that the inequality (41) might be strengthened up to the double inequality (42). This nonetheless conditions the approximate solution to be more strict, underscoring the monotonous character of the support cardinality as the sampling changes within its minimal neighborhood.

Definition 2. The weakly consistent solution (32) of the game (31) is called consistent for being the approximate solution of the game (6) if the inequalities (42) and

$$\max_{q\in\left\{\overline{1,\, Q_1^*(0)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(0),\, \overline{\mathbf{X}}_{q+1}(0)\right) \leqslant \max_{q\in\left\{\overline{1,\, Q_1^*(-1)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q(-1),\, \overline{\mathbf{X}}_{q+1}(-1)\right) \tag{62}$$

and

$$\max_{w\in\left\{\overline{1,\, Q_2^*(0)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(0),\, \overline{\mathbf{Y}}_{w+1}(0)\right) \leqslant \max_{w\in\left\{\overline{1,\, Q_2^*(-1)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w(-1),\, \overline{\mathbf{Y}}_{w+1}(-1)\right) \tag{63}$$

are true. The game (31) solution (32) is called 1-consistent. Every strategy and its support in the 1-consistent solution are called consistent or 1-consistent.

Properly speaking, neither conditions within Definition 1, nor conditions within Definition 2 guarantee the perfection of the game (6) approximation as the game (31) with its (weakly) consistent solution (32). But (weak) 1-consistency signifies that both the optimal payoff and the players' supports differentiate less as the sampling grows up minimally. And this property becomes more strict with the inequalities (42) and (62), (63). Note that in controlling the players' optimal strategies for their weak 1-consistency, there are nine inequalities (40), (41), and (56) — (61) to be checked. And there are 13 inequalities (40), (42), and (56) — (63) to be checked for controlling the players' optimal strategies for their 1-consistency. The following assertion will help to avoid superfluous computations in checking weak 1-consistency.

Theorem 3. If the solution

$$\left\{\left\{p_*(u_1,\, 1)\right\}_{u_1=1}^{Q_1(1)},\, \left\{q_*(u_2,\, 1)\right\}_{u_2=1}^{Q_2(1)}\right\} \tag{64}$$

of 1-matrix game is completely mixed, then for checking weak 1-consistency of the solution (32) it is sufficient to check inequalities (40) and (58) — (61).

Proof. Inasmuch as the situation (64) is completely mixed then

$$Q_1^*(1) = Q_1(1) = \prod_{m=1}^{M}\left(S_m^{\langle 1\rangle} + 2\right) > \prod_{m=1}^{M}\left(S_m^{\langle 1\rangle} + 1\right) \geqslant Q_1^*(0),$$

$$Q_2^*(1) = Q_2(1) = \prod_{n=1}^{N}\left(S_n^{\langle 2\rangle} + 2\right) > \prod_{n=1}^{N}\left(S_n^{\langle 2\rangle} + 1\right) \geqslant Q_2^*(0),$$

giving us the inequality (41), even with strict sign. Further, as the solution (64) is completely mixed then through the index convention (54) and (55) there are the sets $\left\{\bar{\mathbf{X}}_q(1)\right\}_{q=1}^{Q_1^*(1)} = \left\{\mathbf{X}_q(1)\right\}_{q=1}^{Q_1(1)}$ and $\left\{\bar{\mathbf{Y}}_w(1)\right\}_{w=1}^{Q_2^*(1)} = \left\{\mathbf{Y}_w(1)\right\}_{w=1}^{Q_2(1)}$ such that

$$\max_{q\in\left\{\overline{1,\, Q_1^*(1)-1}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(1),\, \bar{\mathbf{X}}_{q+1}(1)\right) = \max_{m=\overline{1,\, M}} \max_{d_m=\overline{1,\, S_m^{\langle 1\rangle}+1}} \left(x_m^{\langle d_m+1\rangle}(1) - x_m^{\langle d_m\rangle}(1)\right) \tag{65}$$

and

$$\max_{w\in\left\{\overline{1,\, Q_2^*(1)-1}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(1),\, \bar{\mathbf{Y}}_{w+1}(1)\right) = \max_{n=\overline{1,\, N}} \max_{d_n=\overline{1,\, S_n^{\langle 2\rangle}+1}} \left(y_n^{\langle d_n+1\rangle}(1) - y_n^{\langle d_n\rangle}(1)\right). \tag{66}$$

Then, due to (37) and (38), have

$$\max_{q\in\left\{\overline{1,\, Q_1^*(1)-1}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(1),\, \bar{\mathbf{X}}_{q+1}(1)\right) \leqslant \max_{m=\overline{1,\, M}} \max_{d_m=\overline{1,\, S_m^{\langle 1\rangle}}} \left(x_m^{\langle d_m+1\rangle} - x_m^{\langle d_m\rangle}\right) \leqslant$$

$$\leqslant \max_{q\in\left\{\overline{1,\, Q_1^*(0)-1}\right\}} \sqrt{\sum_{m=1}^{M}\left(x_m^{\langle \bar{j}_m(q+1,\, 0)\rangle} - x_m^{\langle \bar{j}_m(q,\, 0)\rangle}\right)^2} = \max_{q\in\left\{\overline{1,\, Q_1^*(0)-1}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(0),\, \bar{\mathbf{X}}_{q+1}(0)\right),$$

$$\max_{w\in\left\{\overline{1,\, Q_2^*(1)-1}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(1),\, \bar{\mathbf{Y}}_{w+1}(1)\right) \leqslant \max_{n=\overline{1,\, N}} \max_{d_n=\overline{1,\, S_n^{\langle 2\rangle}}} \left(y_n^{\langle d_n+1\rangle} - y_n^{\langle d_n\rangle}\right) \leqslant$$

$$\leqslant \max_{w\in\left\{\overline{1,\, Q_2^*(0)-1}\right\}} \sqrt{\sum_{n=1}^{N}\left(y_n^{\langle \bar{j}_{M+n}(w+1,\, 0)\rangle} - y_n^{\langle \bar{j}_{M+n}(w,\, 0)\rangle}\right)^2} = \max_{w\in\left\{\overline{1,\, Q_2^*(0)-1}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(0),\, \bar{\mathbf{Y}}_{w+1}(0)\right),$$

giving us the inequalities (56) and (57). The theorem has been proved.

Consistency of finite support optimal strategy, approximating the player's genuine optimal strategy, underscores quality of the approximation. Clearly, 1-consistency can be widened to λ-consistency by $\lambda \in \mathbb{N}$, invoking $2\lambda + 1$ matrix games, approximating the initial one (6).

7. Infinite Antagonistic Game Approximation in λ-Consistency

Definition 3. The solution (32) of the game (31) is called weakly λ-consistent for being the approximate solution of the game (6) if the inequalities

$$\left|v_*(\mu) - v_*(\mu+1)\right| \leqslant \left|v_*(\mu-1) - v_*(\mu)\right|, \tag{67}$$

$$Q_r^*(\mu+1) \geqslant Q_r^*(\mu) \quad \text{by} \quad r \in \{1,\, 2\}, \tag{68}$$

$$\max_{q\in\left\{\overline{1,\, Q_1^*(\mu+1)-1}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(\mu+1),\, \bar{\mathbf{X}}_{q+1}(\mu+1)\right) \leqslant \max_{q\in\left\{\overline{1,\, Q_1^*(\mu)-1}\right\}} \rho_{\mathbb{R}^M}\left(\bar{\mathbf{X}}_q(\mu),\, \bar{\mathbf{X}}_{q+1}(\mu)\right), \tag{69}$$

$$\max_{w\in\left\{\overline{1,\, Q_2^*(\mu+1)-1}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(\mu+1),\, \bar{\mathbf{Y}}_{w+1}(\mu+1)\right) \leqslant \max_{w\in\left\{\overline{1,\, Q_2^*(\mu)-1}\right\}} \rho_{\mathbb{R}^N}\left(\bar{\mathbf{Y}}_w(\mu),\, \bar{\mathbf{Y}}_{w+1}(\mu)\right), \tag{70}$$

$$\max_{H_M^{\langle 1\rangle}} \left|h_1(u_1,\, \mu) - h_1(u_1,\, \mu+1)\right| \leqslant \max_{H_M^{\langle 1\rangle}} \left|h_1(u_1,\, \mu-1) - h_1(u_1,\, \mu)\right|, \tag{71}$$

$$\max_{H_N^{\langle 2\rangle}} \left|h_2(u_2,\, \mu) - h_2(u_2,\, \mu+1)\right| \leqslant \max_{H_N^{\langle 2\rangle}} \left|h_2(u_2,\, \mu-1) - h_2(u_2,\, \mu)\right|, \tag{72}$$

and

$$\left\| h_1\left(u_1,\,\mu\right)-h_1\left(u_1,\,\mu+1\right)\right\| \leqslant \left\| h_1\left(u_1,\,\mu-1\right)-h_1\left(u_1,\,\mu\right)\right\| \text{ in } \mathbb{L}_2\left(H_M^{\langle 1\rangle}\right), \tag{73}$$

$$\left\| h_2\left(u_2,\,\mu\right)-h_2\left(u_2,\,\mu+1\right)\right\| \leqslant \left\| h_2\left(u_2,\,\mu-1\right)-h_2\left(u_2,\,\mu\right)\right\| \text{ in } \mathbb{L}_2\left(H_N^{\langle 2\rangle}\right) \tag{74}$$

are true $\forall\,\mu=\overline{1-\lambda,\,\lambda-1}$ by $\lambda\in\mathbb{N}$. Every strategy and its support in the weakly λ-consistent solution are called weakly λ-consistent.

Definition 4. The weakly λ-consistent solution (32) of the game (31) is called λ-consistent for being the approximate solution of the game (6) if the inequalities

$$Q_r^*\left(\mu\right)\geqslant Q_r^*\left(\mu-1\right) \text{ by } r\in\left\{1,\,2\right\} \tag{75}$$

and

$$\max_{q\in\left\{\overline{1,\,Q_1^*(\mu)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q\left(\mu\right),\,\overline{\mathbf{X}}_{q+1}\left(\mu\right)\right)\leqslant \max_{q\in\left\{\overline{1,\,Q_1^*(\mu-1)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q\left(\mu-1\right),\,\overline{\mathbf{X}}_{q+1}\left(\mu-1\right)\right) \tag{76}$$

and

$$\max_{w\in\left\{\overline{1,\,Q_2^*(\mu)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w\left(\mu\right),\,\overline{\mathbf{Y}}_{w+1}\left(\mu\right)\right)\leqslant \max_{w\in\left\{\overline{1,\,Q_2^*(\mu-1)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w\left(\mu-1\right),\,\overline{\mathbf{Y}}_{w+1}\left(\mu-1\right)\right) \tag{77}$$

are true $\forall\,\mu=\overline{1-\lambda,\,\lambda-1}$ by $\lambda\in\mathbb{N}$. Every strategy and its support in the λ-consistent solution are called λ-consistent.

This conception of λ-consistency is some straightforward, because Definition 4 has been conceived to be stated similarly to Definition 2. However, by $\lambda\in\mathbb{N}\setminus\left\{1\right\}$ there is no need to check all $2\lambda-1$ inequalities in (73) and $2\lambda-1$ inequalities in (74) for each of the players.

Theorem 4. If the inequalities

$$Q_r^*\left(1-\lambda\right)\geqslant Q_r^*\left(-\lambda\right) \text{ by } r\in\left\{1,\,2\right\} \tag{78}$$

and

$$\max_{q\in\left\{\overline{1,\,Q_1^*(1-\lambda)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q\left(1-\lambda\right),\,\overline{\mathbf{X}}_{q+1}\left(1-\lambda\right)\right)\leqslant \max_{q\in\left\{\overline{1,\,Q_1^*(-\lambda)-1}\right\}} \rho_{\mathbb{R}^M}\left(\overline{\mathbf{X}}_q\left(-\lambda\right),\,\overline{\mathbf{X}}_{q+1}\left(-\lambda\right)\right) \tag{79}$$

and

$$\max_{w\in\left\{\overline{1,\,Q_2^*(1-\lambda)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w\left(1-\lambda\right),\,\overline{\mathbf{Y}}_{w+1}\left(1-\lambda\right)\right)\leqslant \max_{w\in\left\{\overline{1,\,Q_2^*(-\lambda)-1}\right\}} \rho_{\mathbb{R}^N}\left(\overline{\mathbf{Y}}_w\left(-\lambda\right),\,\overline{\mathbf{Y}}_{w+1}\left(-\lambda\right)\right) \tag{80}$$

are true then the weakly λ-consistent solution (32) of the game (31) is λ-consistent.

Proof. Inasmuch as the inequalities (68) — (70) are true $\forall\,\mu=\overline{1-\lambda,\,\lambda-1}$ then, having added the inequalities (78) — (80) to them, there are true the inequalities (75) — (77). The theorem has been proved.

Apparently, (weak) $\left(\lambda-1\right)$-consistency follows (weak) λ-consistency. The generalizing Definitions 3 and 4 prescribe the monotonic-like properties for a series of matrix games, obtained after sampling within the assigned neighborhood. The greater λ (the wider this sampling neighborhood) is, the more suitable matrix game (31) for being called the approximation of the infinite game (6).

8. An Example of Consistency Deployment

For exemplifying some practical peculiarities of consistency, try solving approximately the known infinite antagonistic game on the unit square

$$H_1^{\langle 1\rangle}\times H_1^{\langle 2\rangle}=\left[0;\,1\right]\times\left[0;\,1\right]\subset\mathbb{R}^2 \tag{81}$$

as a time selection game [12]. By putting $x = \mathbf{X}$ and $y = \mathbf{Y}$, this game kernel is such [11] that $K(x, y) = 2x - y$ by $x < y$ and $K(x, y) = x - 2y$ by $x > y$, and $K(x, x) = 0$. Unification of these formulas brings us to the whole formula:

$$K(x, y) = (2x - y) \cdot \frac{1 + \operatorname{sign}(y - x)}{2} + (x - 2y) \cdot \frac{1 + \operatorname{sign}(x - y)}{2} = \\ = \frac{3x - 3y}{2} + (x + y) \cdot \frac{\operatorname{sign}(y - x)}{2}. \quad (82)$$

The function (82) is differentiable almost everywhere on (81), except the line $x = y$, whose ordinary Lebesgue measure in $\mathbb{R}^2$ is null. Owing to $K(x, y) = -K(y, x)$ the game with the kernel (82) on the unit square (81) is symmetric. Consequently, the optimal value in this game is equal to zero. The players' optimal strategies, if exist, have to be identical. They might be found analytically, but the corresponding proof is rather intricate [12].

Just adverting to the support cardinality, we see in Figure 1 that 3-consistency by $S = S_1^{\langle 1 \rangle} = S_1^{\langle 2 \rangle}$ for the kernel (82) is impossible over pretty wide range of the numbers (14), where the sets (9) and (10) are identical:

$$D_1^{\langle 1 \rangle}\left(S_1^{\langle 1 \rangle}\right) = \left\{ x_1^{\langle s_1 \rangle} = \frac{s_1 - 1}{S} \right\}_{s_1 = 1}^{S+1} = \left\{ y_1^{\langle s_1 \rangle} = \frac{s_1 - 1}{S} \right\}_{s_1 = 1}^{S+1} = D_n^{\langle 2 \rangle}\left(S_1^{\langle 2 \rangle}\right). \quad (83)$$

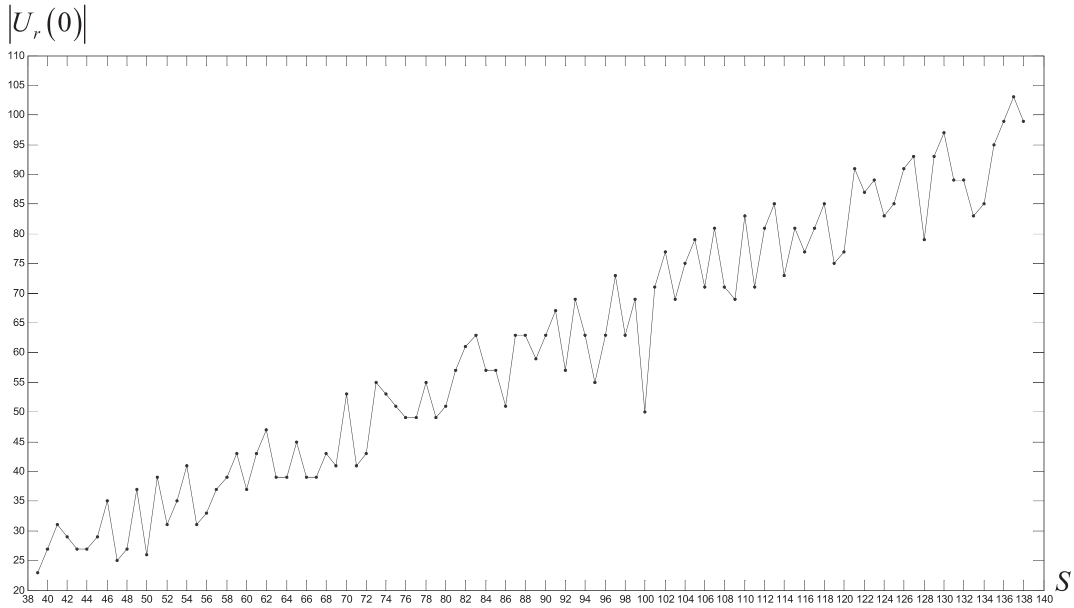

Figure 1. The support cardinality variation by $r \in \{1, 2\}$ and $S = \overline{39, 138}$ for $S = S_1^{\langle 1 \rangle} = S_1^{\langle 2 \rangle}$ in approximating the game with the kernel (82) on the unit square (81)

Here, due to $S = S_1^{\langle 1 \rangle} = S_1^{\langle 2 \rangle}$, every matrix $\mathbf{G}(0) = -\left[\mathbf{G}(0)\right]^{\mathrm{T}}$ and $v_*(0) = 0$, and the players' optimal strategies in the solution (32) are identical $\forall\, S = \overline{39, 138}$. Taking $S = 40$, we can check weak 1-consistency of the solution (Figure 2)

$$\left\{ \left\{ p_*(u_1, 0) \right\}_{u_1 = 1}^{41}, \left\{ q_*(u_2, 0) \right\}_{u_2 = 1}^{41} \right\}. \quad (84)$$

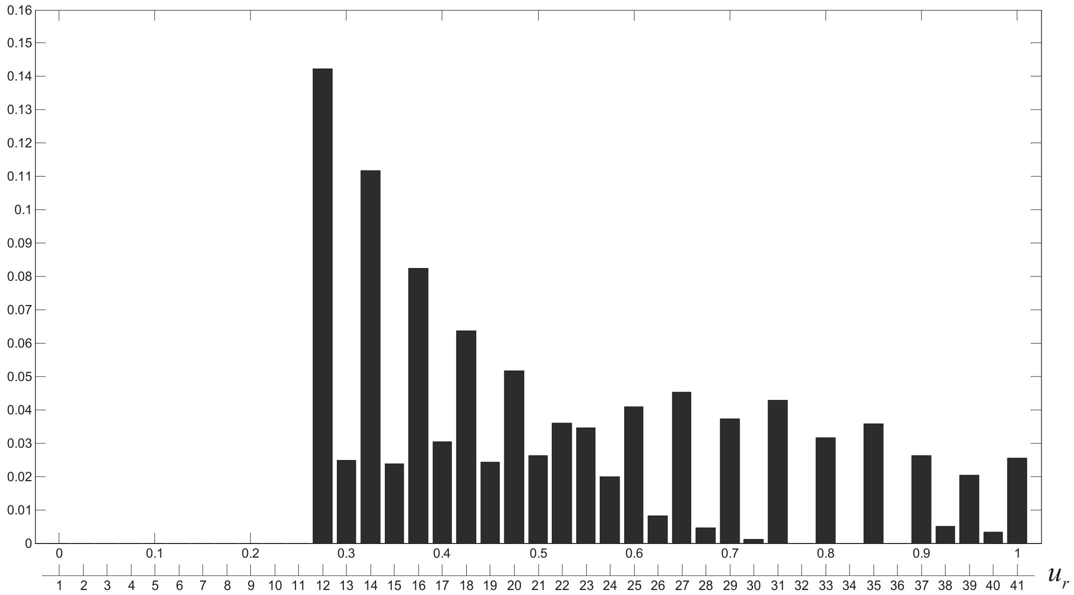

Figure 2. The player's optimal strategy support and its optimal probabilities from the solution (84) for $S = 40$

So, the inequality (40) is already true, and $\left|U_r(1)\right| = 31$, $\left|U_r(0)\right| = 27$, bringing the inequality (41) true also. Figure 3 along with Figure 2 hint that

$$\max_{q \in \{1, 30\}} \rho_{\mathbb{R}}\left(\overline{\mathbf{X}}_q(1), \overline{\mathbf{X}}_{q+1}(1)\right) = \max_{w \in \{1, 30\}} \rho_{\mathbb{R}}\left(\overline{\mathbf{Y}}_w(1), \overline{\mathbf{Y}}_{w+1}(1)\right) = 0,$$

$$\max_{q \in \{1, 26\}} \rho_{\mathbb{R}}\left(\overline{\mathbf{X}}_q(0), \overline{\mathbf{X}}_{q+1}(0)\right) = \max_{w \in \{1, 26\}} \rho_{\mathbb{R}}\left(\overline{\mathbf{Y}}_w(0), \overline{\mathbf{Y}}_{w+1}(0)\right) = 0.05,$$

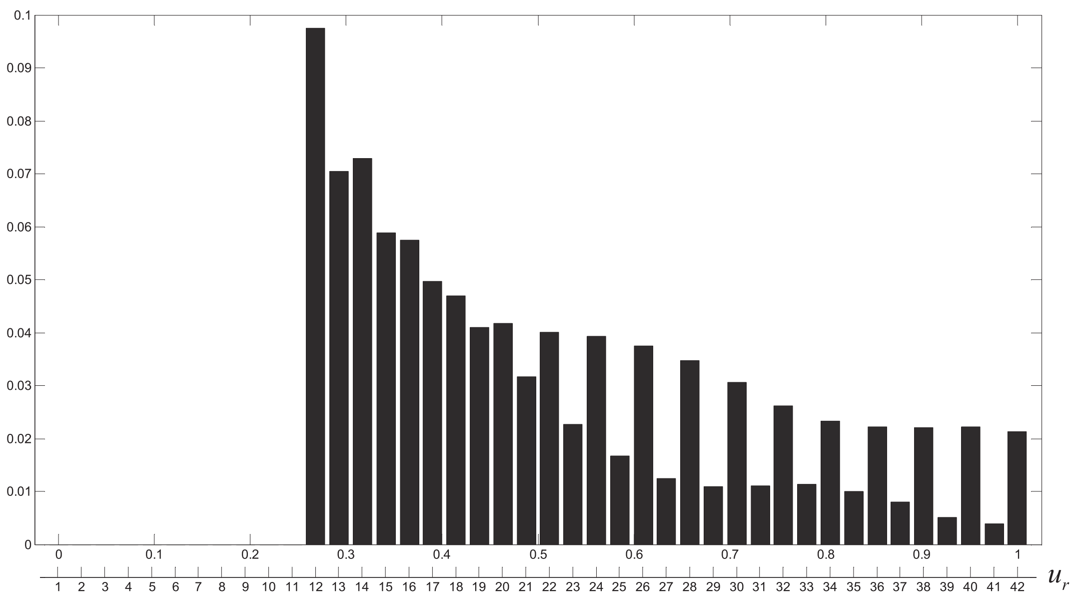

Figure 3. The player's optimal strategy support and its optimal probabilities from the solution $\left\{\left\{p_*(u_1, 1)\right\}_{u_1=1}^{42}, \left\{q_*(u_2, 1)\right\}_{u_2=1}^{42}\right\}$ for $S = 40$

what confirms the inequalities (56) and (57). Hypersurfaces

$$\left\{h_r(u_r, -1), h_r(u_r, 0), h_r(u_r, 1)\right\}_{r=1}^{2} \tag{85}$$

being actually polylines now (Figure 4), bring the inequalities (58) and (59) true on the unit segment $H_1^{\langle 1 \rangle} = H_1^{\langle 2 \rangle} = [0; 1]$ (Figure 5), and the inequalities (60) and (61) hold:

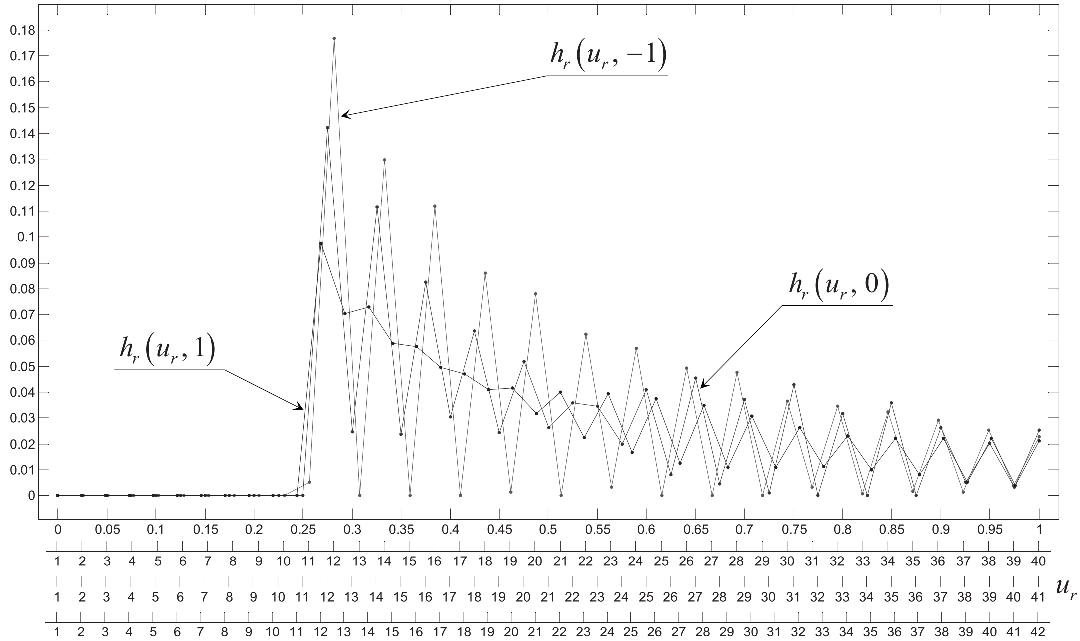

Figure 4. Polylines (85) on the unit segment $H_1^{\langle 1 \rangle} = H_1^{\langle 2 \rangle} = [0; 1]$ for $S = 40$

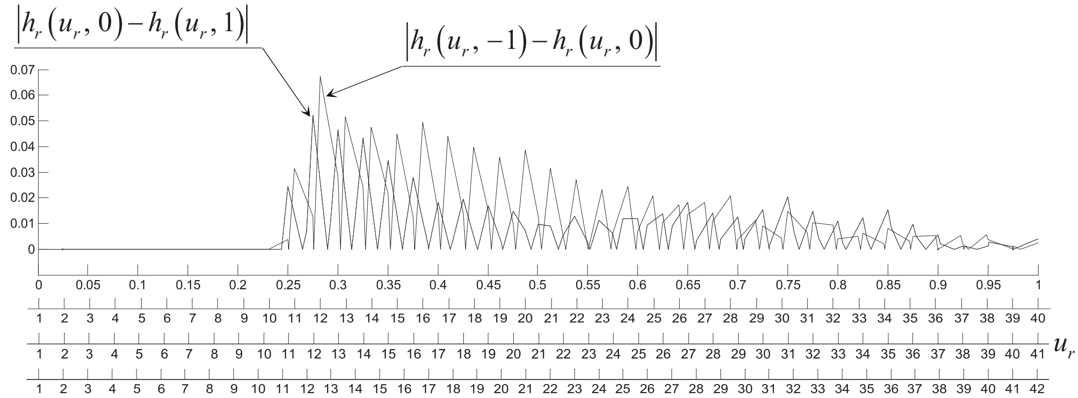

Figure 5. Terms in the inequalities (58) and (59)
on the unit segment $H_1^{\langle 1 \rangle} = H_1^{\langle 2 \rangle} = [0; 1]$ for $S = 40$

$$\left\| h_r(u_r, 0) - h_r(u_r, 1) \right\| < 1.16 \cdot 10^{-4} < 2.74 \cdot 10^{-4} < \left\| h_r(u_r, -1) - h_r(u_r, 0) \right\|$$
$$\text{by } r \in \{1, 2\} \text{ in } \mathbb{L}_2([0; 1]).$$

Therefore, situation (84) is weakly 1-consistent approximate solution of the game with the kernel (82) on the unit square (81). And it is easy to see that this solution is 1-consistent: the inequality (42) is confirmed (Figure 6) with that $|U_r(-1)| = 23$ and

$$\max_{q \in \{1, 22\}} \rho_{\mathbb{R}} \left(\overline{\mathbf{X}}_q(-1), \overline{\mathbf{X}}_{q+1}(-1) \right) = \max_{w \in \{1, 22\}} \rho_{\mathbb{R}} \left(\overline{\mathbf{Y}}_w(-1), \overline{\mathbf{Y}}_{w+1}(-1) \right) = \frac{2}{39} > 0.05,$$

confirming the inequalities (62) and (63).

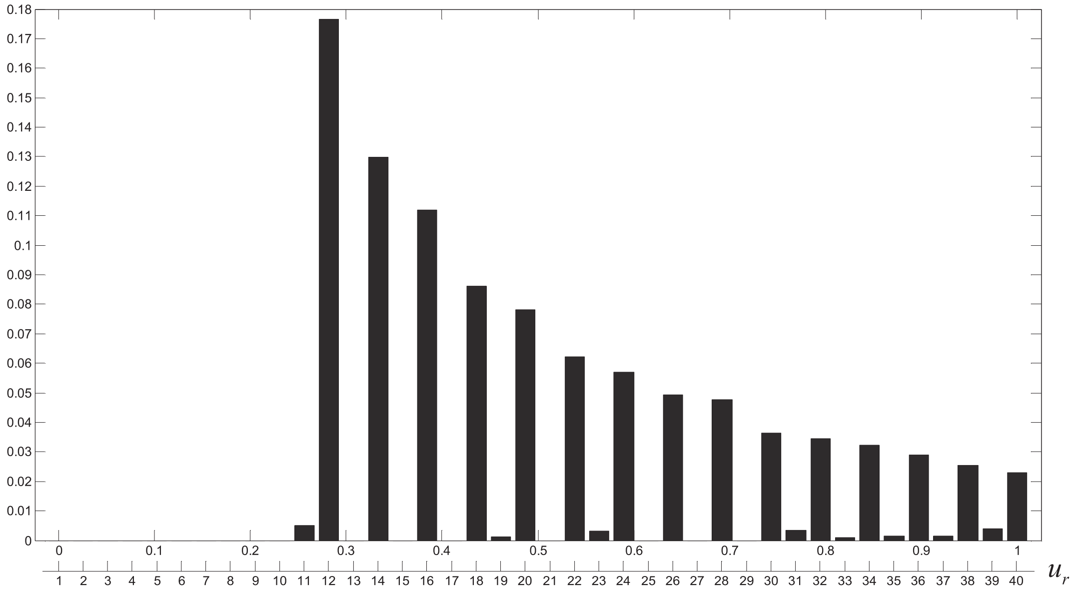

Figure 6. The player's optimal strategy support and its optimal probabilities from the solution $\left\{ \left\{ p_* \left(u_1, -1 \right) \right\}_{u_1=1}^{40}, \left\{ q_* \left(u_2, -1 \right) \right\}_{u_2=1}^{40} \right\}$ for $S = 40$

Moving rightwards, the solution (32) for $S \in \{41, 42\}$ is not weakly 1-consistent. For $S = 43$ it becomes weakly 1-consistent, and it is 1-consistent for $S \in \{44, 45\}$. Father on, leaps of consistency continue (for instance, in local maximums of the polyline in Figure 1). Nevertheless, Figure 7 shows the player's approximate optimal strategy polylines resemble the analytical version of the genuine optimal strategy (for the first player)

$$\hat{p}(x) = \frac{x^{-1.5}}{2} \cdot \left(1 + \operatorname{sign}(x - 0.25) - \operatorname{sign}(|x - 0.25|) \cdot \frac{1 + \operatorname{sign}(x - 0.25)}{2} \right)$$

$$\text{by } x \in H_1^{\langle 1 \rangle} = [0; 1] \tag{86}$$

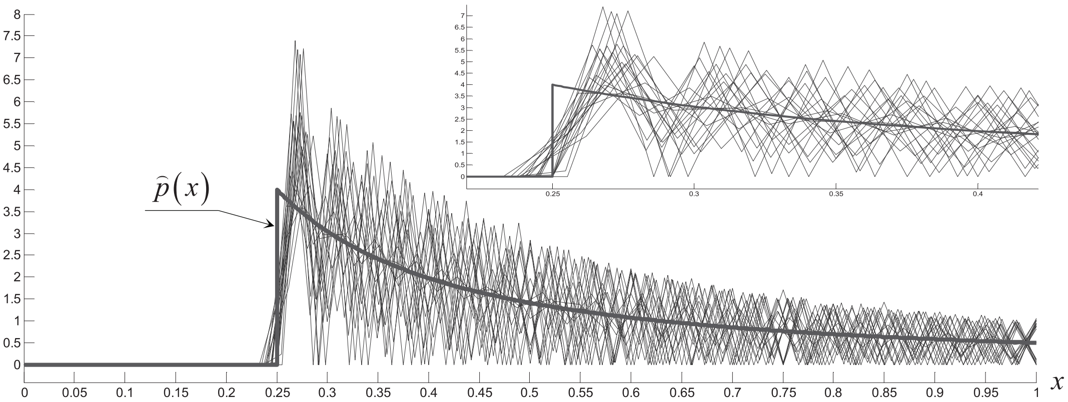

Figure 7. Genuine optimal strategy on the background of the player's optimal strategy polylines obtained for $S = \overline{40, 60}$

deduced in the handbook [11] as $\hat{p}(x) = 0$ by $x \in [0; 0.25)$ and $\hat{p}(x) = \frac{x^{-1.5}}{2}$ by $x \in [0.25; 1]$. This resemblance testifies for expedience of the finite support usage rather than trying to practice the probability density (86). And the sampling with (83) for $S = 40$ is plainly satisfactory owing to the 1-consistent solution (84).

9. Discussible and Completive Notices

The conception of consistency has been intended for proper approximation of infinite antagonistic games. The antagonistic game (6), defined on the unit hypercube (7), is isomorphic to compact games in $\mathbb{R}^{M+N}$. According to the isomorphism, the stated approximation way is available to be applied for solving infinite antagonistic games on compact action spaces.

Surely, if conditions of the (weak) λ-consistency of a strategy support, approximating the unknown genuine strategy, hold true, then it is not sufficient to say that the solution (32) is (weakly) λ-consistent. Speaking strictly, it is unclear whether the player's strategy support, satisfying the conditions of its λ-consistency, causes at least the weak 1-consistency conditions for the other player's strategy support.

Before approximating, the weak 1-consistency ought to be checked first. The check consecution starts with checking the inequalities (41), where the investigator must solve only two matrix games and count the cardinalities of four supports. Then the inequality (40) is recommended to be checked, where (-1)-matrix game is solved already. Then goes subconsecution of checking the inequalities (56) — (61). In checking the weakly consistent solution for its consistency, the inequalities (42) precede the inequalities (62) and (63). Namely the stated consecutions are preferable, because the easiest consistency conditions are checked before the more complicated ones in order to avoid superfluous computations over non-consistent solutions, being revealed after easier comparisons like (41) or, generally, (68). By the way, if (weak) λ-consistency by $\lambda \in \mathbb{N} \backslash \{1\}$ is to be checked then it is better to get started with the inequalities (68) $\forall\ \mu = \overline{1-\lambda,\ \lambda-1}$.

The main lack of the stated approximation way is that there are neither proved the limit

$$\lim_{\delta \to \infty} v_*(\delta)$$

existence and its convergence to the genuine optimal value in the game (6) in the situation approximated by the situation (32), nor proved the limits

$$\lim_{\delta \to \infty} h_r(u_r, \delta) \ \text{ by } \ r \in \{1, 2\}$$

existence and their convergence to the hypersurfaces from the genuine situation approximated by the situation (32). But anyway the represented method of converting the infinite antagonistic game on the unit hypercube into the matrix game lets have the solution to the conflict object, even when the initial game is solved in ε-equilibrium situations or doesn't have solution at all. Besides, the approximate solution with the finite strategy supports is practiced freer, because there are enabled discrete variates and there no continuous variates.

Nevertheless, the assertions proved above are needful and useful. For choosing the numbers (14) and points (17) in order to sample the game (6) kernel, there is Theorem 1, where the hypersurface $K(\mathbf{X}, \mathbf{Y})$ local extremums must be found before. Then the finite game (13) on the finite hypercubic irregular lattice $D^{\langle 1 \rangle} \times D^{\langle 2 \rangle}$ is mapped into the matrix game (31) with the matrix map $\mathbf{P}(0) \to \mathbf{G}(0)$, whose conditions and maintenance of one-to-one indexing are provided in Theorem 2. While checking weak 1-consistency, Theorem 3 quits of superfluous computations over (41), (56), (57) with occurrence of completely mixed 1-matrix game. And finally, Theorem 4 prompts how to check λ-consistency easier by $\lambda \in \mathbb{N} \backslash \{1\}$, using overlapped conditions for weakly λ-consistent solution and λ-consistent solution, leaving only four inequalities (78) — (80) to be checked.

Unlike ε-net-construction technique, sampling the hypersurface $K(\mathbf{X}, \mathbf{Y})$ can be done with numbers (14) and finite sets (9) — (12) even if the game kernel is unbounded and non-measurable and non-differentiable. Irregularity in sampling ensures free choice of the sampling points along every dimension, where the only restriction is the conditions of

Theorem 1. Accurateness of the sampling is either strengthened with increasing the sampling numbers (14) or underscored with consistency of higher rank.

The stated method of approximating infinite antagonistic games, grounded on the irregular sampling and consistency control, could be adapted to solving approximately noncooperative games. Then the suggested free choice of the sampling points along every dimension will appear useful again because multidimensional matrices of players' payoff values are sometimes better to have them in square or hypercubic format [12, 11, 4] (having identical size of each dimension). However, the main problem is to sort sets $\left\{\mathbf{X}_q\left(\delta\right)\right\}_{q=1}^{Q_1^*(\delta)}$ and $\left\{\mathbf{Y}_w\left(\delta\right)\right\}_{w=1}^{Q_2^*(\delta)}$ into sets $\left\{\bar{\mathbf{X}}_q\left(\delta\right)\right\}_{q=1}^{Q_1^*(\delta)}$ and $\left\{\bar{\mathbf{Y}}_w\left(\delta\right)\right\}_{w=1}^{Q_2^*(\delta)}$ effectively. This problem is combinatorial and its solution isn't trivial, unless the strategy is completely mixed. Another open question is that the result of the sorting depends on selection of the initial points $\bar{\mathbf{X}}_1\left(\delta\right)\in\left\{\mathbf{X}_q\left(\delta\right)\right\}_{q=1}^{Q_1^*(\delta)}$ and $\bar{\mathbf{Y}}_1\left(\delta\right)\in\left\{\mathbf{Y}_w\left(\delta\right)\right\}_{w=1}^{Q_2^*(\delta)}$, whereas the criterion for this selection remains uncertain.

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