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Interpretation of Fuzzy Attribute Subsets in Generalized One-Sided Concept Lattices

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Abstract

In this paper we describe possible interpretation and reduction of fuzzy attributes in Generalized One-sided Concept Lattices (GOSCL). This type of concept lattices represent generalization of Formal Concept Analysis (FCA) suitable for analysis of data tables with different types of attributes. FCA as well as generalized one-sided concept lattices represent conceptual data mining methods. With growing number of attributes the interpretation of fuzzy subsets may become unclear, hence another interpretation of this fuzzy attribute subsets can be valuable. The originality of the presented method is based on the usage of one-sided concept lattices derived from submodels of former object-attribute model by grouping attributes with the same truth value structure. This leads to new method for attribute reduction in GOSCL environment.

Keywords: Generalized one-sided concept lattices, Galois connections, object-attribute model

1. Introduction

The large amount of available data and the growing needs for their analysis brings up the new challenges to the area of data mining. The challenges include capture, curation, storage, search, sharing, analysis, and visualization. It is an emerging field where the need for more effective and understandable methods and algorithms is evident.

Formal concept analysis (FCA, cf. [8]) is one of the effective mathematical tools for identification of conceptual structures among data sets and knowledge processing. Mathematical theory of FCA is based on the notions of formal context, which formally describes object-attribute model and formal concept, which is formed by extent intent pair. The extent is understood as the collection of all objects belonging to the concept, while the intent is the multitude of all attributes common to all those objects. The set of all formal concepts of a formal context forms a complete lattice, called the concept lattice, which reflects the relationship of generalization and specialization among the formal concepts. A concept lattice is an effective tool in FCA and is very suitable for mining potential concepts of datasets. Nowadays, FCA has been applied to a variety of fields such as decision making, information retrieval, data mining, knowledge discovery, software engineering, business intelligence, as well as in other areas related to machine learning and artificial intelligence.

In the classical formal contexts, the relationship between the objects and the attributes is described by a binary relation that can only specify whether or not an attribute is possessed by an object. In many real applications, however, the relationship may be many-valued or fuzzy. Therefore, some attempts have recently been devoted to introduce fuzzy concept lattice with properties similar with the classical ones. We mention the general approaches dealing with fuzzy subsets of objects and fuzzy subsets of attributes, cf., [1],[2],[12],[20],[21] and [22]. All interesting approaches are extensively studied and extended in order to achieve their application for different types of data analytical problems and inputs, e.g., we can mention many new results related to different fuzzy models and related topics, cf., [3],[4],[13],[15],[16],[19].

A special role in fuzzy FCA play one-sided concept lattices, where usually objects are considered as a crisp subsets and attributes obtain fuzzy values. In this case interpretation of object clusters is straightforward as in classical FCA. Consequently, all known applications developed for classical concept lattices can be used in the theory of one-sided concept lattices. From existing one-sided approaches we mention papers of Krajči [11], Yahia and Jaoua [5], [10], All these mentioned approaches allow to consider only one type of structure for truth degrees. In [21] and [6] it was described an approach to one-sided concept lattices involving different types of truth value structures, which generalizes all recently known approaches and is more convenient for analysis of object-attribute models. An application of this approach to the object-attribute models with different types of attributes can be found in [7].

Since mathematical theory of concept lattices is based on the notion of Galois connection, in preliminary section we recall some basic notations and results from theory of partially ordered sets and Galois connections. Further, we give a definition of formal context, which is mathematical formalization of the object-attribute models with different types of attributes. Next we provide theoretical background for generalized one-sided concept lattices, which are suitable for analysis of such object-attribute models. We also present an incremental algorithm for generation of generalized one-sided concept lattices from the given formal context.

In many real situations the number of considered objects and attributes rise dramatically. In this case the corresponding generalized one-sided concept lattice become too large and readibility of hierarchical relations contained in concept lattice becomes difficult. Hence the reduction of attribute set can help with readibility of informations contained in hierarchical structure of concept lattice. In the third section we describe one of the possible reduction of attribute sets, using submodels of original object-attribute model. An important aspect of our reduction is, that this reduction carries all information as former object-attribute model. In mathematical language we describe a construction of formal context with fever attributes, such that corresponding generalized one-sided concept lattices are isomorphic. Our reduction involves one-sided concept lattices with one type of truth value structure. This theory is well developed and can be usefully applied in many real situations dealing with object-attribute models with one type of attribute. We prove two theorems which show the isomorphism between generalized one-sided concept lattice derived from original formal context and concept lattice derived from reduced one. At the end we demonstrate this reduction on simple illustrative example, where we also show an interpretation of object clusters by another object clusters obtained from reduced formal context.

2. Preliminary section

In this section we describe the necessary theoretical background for generalized one-sided concept latices. Basic idea behind fuzzification of concept lattices is the usage of graded truth. Classical logic is bivalent, i.e., each proposition is either true or false. In fuzzy logic, to each proposition there is assigned a truth degree from some scale of truth degrees with more than two values.

Generalized one-sided concept lattices as they are presented in [21] and [6] are generalizations of already known one-sided concept lattices. They are build in the framework of crisp subsets of objects and generalized L-fuzzy subsets of attributes. Classical L-fuzzy subsets described by Goguen [9] are represented as functions

$$f\colon U \to L \tag{1}$$

from some universe U into truth value structure L, which is represented by complete lattice. The set of all L-fuzzy subsets is usually denoted by L^U. Let us remark that by complete lattice we understand partially ordered set $(L, \leq)$ where least upper bound or supremum of any subset $H \subseteq L$ exists in L and dually greatest lower bound or infimum of H exists in L. We shall denote the supremum of the subset $H \subseteq L$ by symbol $\bigvee H$ and infimum of H by symbol $\bigwedge H$.

As a generalization of the notion of L-fuzzy subset we will deal with so-called generalized fuzzy sets. This notion appeared in connection with fuzzy formal concept analysis for the first time in [21]. Let $U \neq \emptyset$ be an universe and $(L_u)_{u\in U}$ be a system of possibly different complete lattices, which represent truth value structure for each element $u \in U$. By a generalized fuzzy set we will consider a function

$$f\colon U \to \bigcup_{u\in U} L_u \tag{2}$$

such that $f(u) \in L_u$ for each $u \in U$. Moreover we will consider "componentwise" partial order on the set of all generalized fuzzy subsets, i.e., $f \leq g$ if $f(u) \leq g(u)$ for all $u \in U$. From the algebraic point of view the concept of the set of generalized fuzzy subsets coincides with the notion of direct product of complete lattices $(L_u)_{u\in U}$, which is usually denoted by the symbol $\prod_{u\in U} L_u$. In the sequel we will use the same symbol for the set of generalized fuzzy subsets of the system $(L_u)_{u\in U}$. If we consider the system $(L_u)_{u\in U}$ of identical lattices, i.e., $L_u = L$ for all $u \in U$, then we obtain the equality $L^U = \prod_{u\in U} L_u$. Hence, the notion of generalized fuzzy subsets or direct product of lattices generalizes the notion of fuzzy subsets. It is well known fact that the direct product of lattices forms complete lattice if and only if all members of the family are complete lattices. The straightforward computations show that the lattice operations in the direct product $\prod_{u\in U} L_u$ of complete lattices are calculated componentwise, i.e., for any subset $\{f_j : j \in J\} \subseteq \prod_{u\in U} L_u$ we obtain

$$\Big(\bigvee_{j\in J} f_j\Big)(u) = \bigvee_{j\in J} f_j(u) \quad \text{and} \quad \Big(\bigwedge_{j\in J} f_j\Big)(u) = \bigwedge_{j\in J} f_j(u), \tag{3}$$

where this equalities hold for each index $u \in U$.

If one consider two element lattice $\mathbf{2} = \{0, 1\}$ with $0 < 1$, then $\mathbf{2}^U$ represents classical or crisp subsets of the set U. In this case $\mathbf{2}^U$ is identified with characteristic functions of the set U, thus $\mathbf{2}^U$ represents the power set of the set U.

Theory of fuzzy concept lattices is based on pairs of mappings between complete lattices, commonly known as Galois connections. Let $(P, \leq)$ and $(Q, \leq)$ be an ordered sets and let

$$\varphi\colon P \to Q \quad \text{and} \quad \psi\colon Q \to P$$

be maps between these ordered sets. Such a pair (φ, ψ) of mappings is called a Galois connection between the ordered sets if the following condition is fulfilled:

$$p \leq \psi(q) \quad \text{if and only if} \quad \varphi(p) \geq q. \tag{4}$$

In order to describe generalized one-sided concept lattices, we first formalize notion of object-attribute model with different types of attributes. As it is usual in classical FCA we will call this formalization as generalized one-sided formal context.

A 4-tuple $\big(B, A, \mathcal{L}, R\big)$ is said to be a generalized one-sided formal context if the following conditions are fulfilled:

(1) B is a non-empty set of objects and A is a non-empty set of attributes.

(2) $\mathcal{L}\colon A \to \mathsf{CL}$ is a mapping from the set of attributes to the class of all complete lattices. Hence, for any attribute a, $\mathcal{L}(a)$ denotes the complete lattice, which represents structure of truth values for attribute a.

(3) R is generalized incidence relation, i.e., $R(b,a) \in \mathcal{L}(a)$ for all $b \in B$ and $a \in A$. Thus, $R(b,a)$ represents a degree from the structure $\mathcal{L}(a)$ in which the element $b \in B$ has the attribute a.

In theory of one-sided concept lattices the main aim is to introduce a Galois connection between classical subsets of the set of all objects $\mathbf{2}^B$ and the direct products of complete lattices $\prod_{a\in A}\mathcal{L}(a)$ which represents a generalization of fuzzy subsets of the attribute universe A.

Let $\big(B, A, \mathcal{L}, R\big)$ be a generalized one-sided formal context. Then we define a pair of mapping $\uparrow\colon \mathbf{2}^B \to \prod_{a\in A}\mathcal{L}(a)$ and $\downarrow\colon \prod_{a\in A}\mathcal{L}(a) \to \mathbf{2}^B$ as follows:

$$\uparrow\big(X\big)(a) = \bigwedge_{b\in X} R(b,a), \tag{5}$$

$$\downarrow(g) = \{b \in B:\ \forall a \in A,\ g(a) \le R(b,a)\}. \tag{6}$$

For any generalized one-sided formal context the pair $(\uparrow,\downarrow)$ defined by (5) and (6) forms a Galois connection between $\mathbf{2}^B$ and $\prod_{a\in A}\mathcal{L}(a)$.

Based on this Galois connection we are able to define one-sided concept lattices. For formal context $\big(B, A, \mathcal{L}, R\big)$ denote $\mathcal{C}\big(B, A, \mathcal{L}, R\big)$ the set of all pairs (X, g), where $X \subseteq B$, $g \in \prod_{a\in A}\mathcal{L}(a)$, satisfying

$$\uparrow\big(X\big) = g \quad \text{and} \quad \downarrow(g) = X.$$

Set X is usually referred as *extent* and g as *intent* of the concept (X, g).

Further we define partial order on $\mathcal{C}\big(B, A, \mathcal{L}, R\big)$ as follows:

$$(X_1, g_1) \le (X_2, g_2) \quad \text{iff} \quad X_1 \subseteq X_2 \text{ iff } g_1 \ge g_2. \tag{7}$$

Let $\big(B, A, \mathcal{L}, R\big)$ be a generalized one-sided formal context. Then $\mathcal{C}\big(B, A, \mathcal{L}, R\big)$ with the partial order defined by rule (7) forms a complete lattice, where

$$\bigwedge_{i\in I}(X_i, g_i) = \Big(\bigcap_{i\in I} X_i, \uparrow\downarrow\big(\bigvee_{i\in I} g_i\big)\Big) \tag{8}$$

$$\bigvee_{i\in I}(X_i, g_i) = \Big(\downarrow\uparrow\big(\bigcup_{i\in I} X_i\big), \bigwedge_{i\in I} g_i\Big) \tag{9}$$

for each family $(X_i, g_i)_{i\in I}$ of elements from $\mathcal{C}\big(B, A, \mathcal{L}, R\big)$.

At the end of this section we recall an algorithm (see Algorithm 1) for creation of generalized one-sided concept lattice. The main idea is to create set of all intents first and consequently using (6) obtain the corresponding set of pairs (extent, intent).

3. Attribute reduction and representation in GOSCL

In this section we describe possible reduction and representation of the attributes in generalized one-sided lattice framework. Attribute reduction in concept lattice theory was investigated by several authors, cf. [17][18] for attribute reduction in multi-adjoint concept lattices framework and [14][23] for attribute reduction in other specific fuzzy concept lattices. For GOSCL model, our method (as it will be proved by theorems in this section) presents a novel approach which can be successfully applied to object-attribute models with different types of attributes.

Algorithm 1 Incremental algorithm for GOSCL

Require: generalized context $(B, A, \mathcal{L}, R)$
Ensure: set of all concepts $\mathcal{C}(B, A, \mathcal{L}, R)$

$L \leftarrow \prod_{a \in A} \mathcal{L}(a)$ ▷ Direct product of attribute lattices
$I \leftarrow \{1_L\}$ ▷ $I \subseteq L$ will denote set of intents
$\mathcal{C}(B, A, \mathcal{L}, R) \leftarrow \emptyset$
for all $b \in B$ **do**
  $I^* \leftarrow I$ ▷ I^* represents “old" set of intents
  for all $g \in I^*$ **do**
    $I \leftarrow I \cup \{R(b) \wedge g\}$ ▷ Generation of new intent
  end for
end for
for all $g \in I$ **do**
  $\mathcal{C}(B, A, \mathcal{L}, R) \leftarrow \mathcal{C}(B, A, \mathcal{L}, R) \cup \{(\downarrow(g), g)\}$
end for
return $\mathcal{C}(B, A, \mathcal{L}, R)$ ▷ Output of the algorithm

In many real situations large amount of attributes in data tables is represented by the same truth value structure, e.g., there are binary attributes represented by two element chain $\mathbf{2}$ or real valued attributes with truth structure $[0, 1]$. Consider an object-attribute model. which is formally represented by generalized one-sided context $(B, A, \mathcal{L}, R)$. We are able to group attribute with the same truth value structures, i.e., we have partition $\{A_i\}_{i \in I}$ of attribute set such that $A = \bigcup_{i \in I} A_i$ and $A_{i_1} \cap A_{i_2} = \emptyset$ for each $i_1, i_2 \in I$. Let $i \in I$ be any index, then $\mathcal{L}(a_1) = \mathcal{L}(a_2)$ for all $a_1, a_2 \in A_i$. Hence, we will denote the same truth value structure corresponding to each attribute from the set A_i by symbol $\mathcal{L}_i$.

From the former context $(B, A, \mathcal{L}, R)$ we can obtain family of subcontexts $(B, A_i, \mathcal{L}, R_i)$ such that all attributes have assigned the same complete lattice $\mathcal{L}_i$. All these subcontexts now form classical one-sided contexts and we can use well developed theory of classical one-sided concept lattices for representation of fuzzy subsets of attributes. Representation and meaning of single attribute in object-attribute model is given by circumstances under which this model is considered. However, interpretation of more attributes can often become problematic. Thus, considering about subsets of attributes in the framework of the well known classical one-sided concept lattices can be useful for interpretation of results obtained by GOSCL.

First we describe isomorphic representation of generalized one-sided concept lattices with the smaller number of attributes, which are represented by one complex attribute with L-fuzzy subsets as truth value structure. We will use the algebraic fact that $\prod_{a \in A} \mathcal{L}(a) \cong \prod_{i \in I} {\mathcal{L}_i}^{A_i}$, i.e., direct product of complete lattices is isomorphic to direct product of corresponding direct powers. Let $g \in \prod_{a \in A} \mathcal{L}(a)$. We will use the isomorphism given by the rule

$$g \mapsto \overline{g}, \quad \text{where} \quad \overline{g} \in \prod_{i \in I} {\mathcal{L}_i}^{A_i} \tag{10}$$

such that for each $i \in I$, $\overline{g}(i)$ is function

$$\overline{g}(i) \colon A_i \to \mathcal{L}_i \text{ with value } \big(\overline{g}(i)\big)(a) = g(a) \text{ for all } a \in A. \tag{11}$$

Now we define generalized one-sided context with $|I|$ complex attributes. The object set B remains unchanged. Further, we have new attribute set $\overline{A} = \{\overline{A_i} : i \in I\}$, where each symbol $\overline{A_i}$ represents complex attribute derived from set A_i. For all $i \in I$ we put $\mathcal{L}(\overline{A_i}) = \mathcal{L}_i^{A_i}$, thus any complex attribute will be characterized by $\mathcal{L}_i$-fuzzy subsets over universe A_i. Finally, we define the generalized incidence relation $\overline{R} : B \times A \to \bigcup_{i\in I} \mathcal{L}_i^{A_i}$ as follows:

$$\overline{R}(b, \overline{A_i}) = \overline{g}(i), \quad \text{where } \big(\overline{g}(i)\big)(a) = R(b,a) \text{ for each } i \in I, a \in A_i. \tag{12}$$

The 4-tuple $(B, \overline{A}, \mathcal{L}, \overline{R})$ forms a generalized one-sided context. Consequently, we can apply the definition (5) and (6) in order to obtain generalized one-sided concept lattice. We will denote the corresponding mappings by $\overline{\uparrow}$ and $\overline{\downarrow}$. In this case

$$\overline{\uparrow}(X)(\overline{A_i}) = \bigwedge_{b\in X} \overline{R}(b, \overline{A_i}) \tag{13}$$

$$\overline{\downarrow}(\overline{g}) = \{b \in B : \ \forall i \in I, \ \overline{g}(i) \leq \overline{R}(b, \overline{A_i})\} \tag{14}$$

Now we can prove the following theorem, which says that concept lattice derived from former formal context and concept lattice obtained using pair of mappings $\overline{\uparrow}$ and $\overline{\downarrow}$ are isomorphic as partially ordered sets.

Theorem 1 *Let $(B, A, \mathcal{L}, R)$ be a generalized one-sided formal context and $\{A_i\}_{i\in I}$ be partition of attribute set with $\mathcal{L}(a_1) = \mathcal{L}(a_2)$ for each $a_1, a_2 \in A_i$. Then generalized one-sided concept lattices obtained by (5), (6) and (13), (14) respectively, are isomorphic*

Proof We will show that one can find the desired isomorphism by rule

$$(X, g) \mapsto (X, \overline{g}),$$

where $\overline{g}$ is defined using (10) and (11).

First we will show that this correspondence is defined correctly. For this reason we show that for each concept (X, g) the image $(X, \overline{g})$ belongs to the concept lattice $\mathcal{C}(B, \overline{A}, \mathcal{L}, \overline{R})$, i.e. we show that $\uparrow(X) = g$ and $\downarrow(g) = X$ implies $\overline{\uparrow}(X) = \overline{g}$ and $\overline{\downarrow}(\overline{g}) = X$. From definition (5) we have

$$g(a) = \uparrow(X)(a) = \bigwedge_{b\in X} R(b,a),$$

which yields $\big(\overline{g}(i)\big)(a) = \bigwedge_{b\in X} R(b,a)$ by (11). Now applying the rule (13) we obtain

$$\big(\overline{\uparrow}(X)(\overline{A_i})\big)(a) = \big(\bigwedge_{b\in X} \overline{R}(b, \overline{A_i})\big)(a) = \bigwedge_{b\in X} R(b,a),$$

which gives $\overline{\uparrow}(X) = \overline{g}$. Similarly, we can prove

$$\{b \in B : \ \forall a \in A, \ g(a) \leq R(b,a)\} = \{b \in B : \ \forall i \in I, \ \overline{g}(i) \leq \overline{R}(b, \overline{A_i})\}$$

which is equivalent to $\downarrow(g) = \overline{\downarrow}(\overline{g})$.

Finally, we prove that our mapping is bijective order preserving. Since $(X_1, g_1) \leq (X_2, g_2)$ if and only if $(X_1, \overline{g_1}) \leq (X_2, \overline{g_2})$ we obtain that our mapping is order preserving and injective. concept lattices are uniquely determined by the set of extents and our mapping leave extents unchanged, we obtain that resulting structure are isomorphic. If $(X, \overline{g})$ is concept in $\mathcal{C}(B, \overline{A}, \mathcal{L}, \overline{R})$, then $(X, \uparrow(X))$ is the concept in $\mathcal{C}(B, A, \mathcal{L}, R)$ and we have $(X, \overline{\uparrow(X)}) = (X, \overline{g})$, hence this correspondence is surjective too.

■

This isomorphism theorem gives us the possibility to reduce $\mathcal{L}_i$-fuzzy subsets of attributes to the much smaller structures. The main aim is to use corresponding one-sided concept lattices as new truth value structures for attributes $\overline{A}_i$. Let $(B, A_i, \mathcal{L}, R_i)$ be formal contexts obtained by grouping the attributes with same truth value structure $\mathcal{L}_i$ and $\mathcal{C}_i = \mathcal{C}(B, A_i, \mathcal{L}, R_i)$ be the corresponding one-sided concept lattices. Now we define one-sided formal context as follows: The set of objects will be identical with the former one, attribute set will be $\overline{A} = \{\overline{A}_i : i \in I\}$ and the function $\mathcal{L}\colon \overline{A} \to \mathsf{CL}$ is defined as $\mathcal{L}(\overline{A}_i) = \mathcal{C}_i$. The incidence relation $\ddot{R}\colon B \times \overline{A} \to \bigcup_{i\in I} \mathcal{C}_i$ is set to be

$$\ddot{R}(b, \overline{A}_i) = (\downarrow_i \uparrow_i (b), \uparrow_i (b)); \quad \text{for all } b \in B, i \in I, \tag{15}$$

where $(\uparrow_i, \downarrow_i)$ denotes Galois connection obtained from context $(B, A_i, \mathcal{L}, R_i)$ using (5) and (6)

Now we define pair of mapping $\nearrow\colon \mathbf{2}^B \to \prod_{i\in I} \mathcal{C}_i$ and $\swarrow\colon \prod_{i\in I} \mathcal{C}_i \to \mathbf{2}^B$.

$$\nearrow (X)(\overline{A}_i) = \bigvee_{b\in X} \ddot{R}(b, \overline{A}_i) \tag{16}$$

$$\swarrow (g) = \{b \in B : \forall\, i \in I,\ g(i) \geq \ddot{R}(b, \overline{A}_i)\} \tag{17}$$

Similarly as in previous case we define concepts as pairs (X, g), $X \subseteq B$, $g \in \prod_{i\in I} \mathcal{C}_i$ satisfying

$$\nearrow (X) = g \quad \text{and} \quad \swarrow g = X.$$

Further, we define partial order on the set of all concepts as

$$(X_1, g_1) \leq (X_2, g_2) \quad \text{iff} \quad X_1 \subseteq X_2 \quad \text{iff} \quad g_1 \leq g_2.$$

The set of all concepts with this partial order will be denoted by $\mathcal{B}(B, \overline{A}, \mathcal{L}, \ddot{R})$

Theorem 2 *The concept lattice $\mathcal{C}(B, A, \mathcal{L}, R)$ and $\mathcal{B}(B, \overline{A}, \mathcal{L}, \ddot{R})$ are isomorphic.*

Proof First observe that using expression (9) for supremum in concept lattice for any $X \subseteq B$ and $i \in I$ we obtain

$$\nearrow (X)(A_i) = \bigvee_{b\in X} \ddot{R}(b, \overline{A}_i) = \bigvee_{b\in X} (\downarrow_i\uparrow_i (b), \uparrow_i b) =$$

$$\big(\downarrow_i\uparrow_i \big(\bigcup_{b\in X} \downarrow_i\uparrow_i (b)\big), \bigwedge_{b\in X} \uparrow_i (b)\big) = \big(\downarrow_i\uparrow_i (X), \uparrow_i (X)\big).$$

Further, from the definition (5) of $\uparrow_i$ we obtain

$$\uparrow_i (X)(a) = \bigwedge_{b\in X} R(b, a) = \uparrow(X)(a).$$

This yields that mapping $\uparrow$ coincide with each mapping $\uparrow_i$ on the set A_i. Hence we can define a mapping $\Phi\colon \mathcal{C}(B, A, \mathcal{L}, R) \to \mathcal{B}(B, \overline{A}, \mathcal{L}, \ddot{R})$ as

$$\Phi(\uparrow (X))(i) = \uparrow (X) \restriction_{A_i} = \uparrow_i (X).$$

Since the concept lattices are uniquely determined by the set of all intents and Φ is the bijection between the sets of intents, we obtain that Φ is isomorphism between corresponding concept lattices. ■

Next we demonstrate the potential usage of this theorem on complex object-attribute models with different types of attributes. First step is the grouping the particular attributes with the same truth value structures as it is schematically described at Figure 1. The attributes with same truth value structures can be considered more similar each other, hence it is appropriate to consider this group of attributes as sub-object-attribute model of the former one.

Consider the following example.

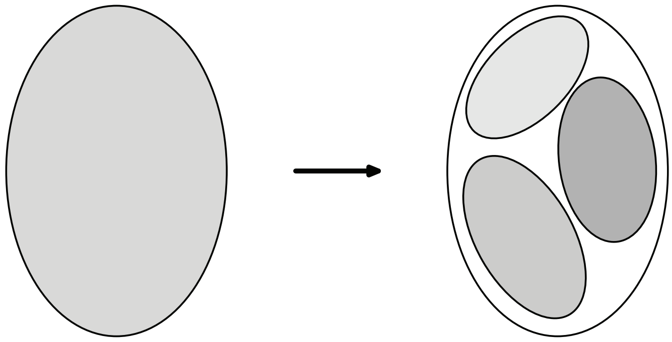

Figure 1: Grouping attributes with same truth value structures.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
r	0	0.2	1	0.3	1	0	0.1	1	2	0
s	1	0.6	0	0.6	0	1	0.5	1	2	1
t	1	1.0	0	0.7	0	0	0.5	3	2	1
u	0	0.2	0	0.3	1	0	0.1	3	1	0
v	1	0.2	3	0.0	0	1	1.0	2	0	1
x	0	0.0	1	1.0	0	1	0.0	2	2	1
y	1	0.2	0	0.3	1	0	0.5	2	2	0
z	0	0.6	0	0.6	1	0	0.1	3	3	1

Table 1: Data table of object-attribute model

Example 1

The object-attribute model, given by Table 1, contains set of objects $B = \{r, s, t, u, v, x, y, z\}$ and set of attributes $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\}$. The attributes are characterized by three different truth value structures. Attributes a_1, a_5, a_6, a_{10} are binary, i.e., $\mathcal{L}(a_1) = \mathcal{L}(a_5) = \mathcal{L}(a_6) = \mathcal{L}(a_{10}) = \mathbf{2}$, where $\mathbf{2}$ represents two element chain. Attributes a_2, a_4, a_7 real, i.e., they are characterized by unit interval $[0, 1]$ of real numbers. Finally, attributes a_3, a_8, a_9 are characterized by four element chain $\mathbf{4} = \{0, 1, 2, 3\}$ with $0 < 1 < 2 < 3$.

By grouping attributes with same truth value structure we obtain three subcontexts depicted in Table 2, corresponding one-sided concept lattices are shown on Figure 2.

	a_1	a_5	a_6	a_{10}
r	0	1	0	0
s	1	0	1	1
t	1	0	0	1
u	0	1	0	0
v	1	0	1	1
x	0	0	1	1
y	1	1	0	0
z	0	1	0	1

	a_2	a_4	a_7
r	0.2	0.3	0.1
s	0.6	0.6	0.5
t	1.0	0.7	0.5
u	0.2	0.3	0.1
v	0.2	0.0	1.0
x	0.0	1.0	0.0
y	0.2	0.3	0.5
z	0.6	0.6	0.1

	a_3	a_8	a_9
r	1	1	2
s	0	1	2
t	0	3	2
u	0	3	1
v	3	2	0
x	1	2	2
y	0	2	2
z	0	3	3

Table 2: Subcontexts for particular types of truth value structures - left: binary attributes, mid: real attributes, right: $\mathbf{4}$-valued attributes

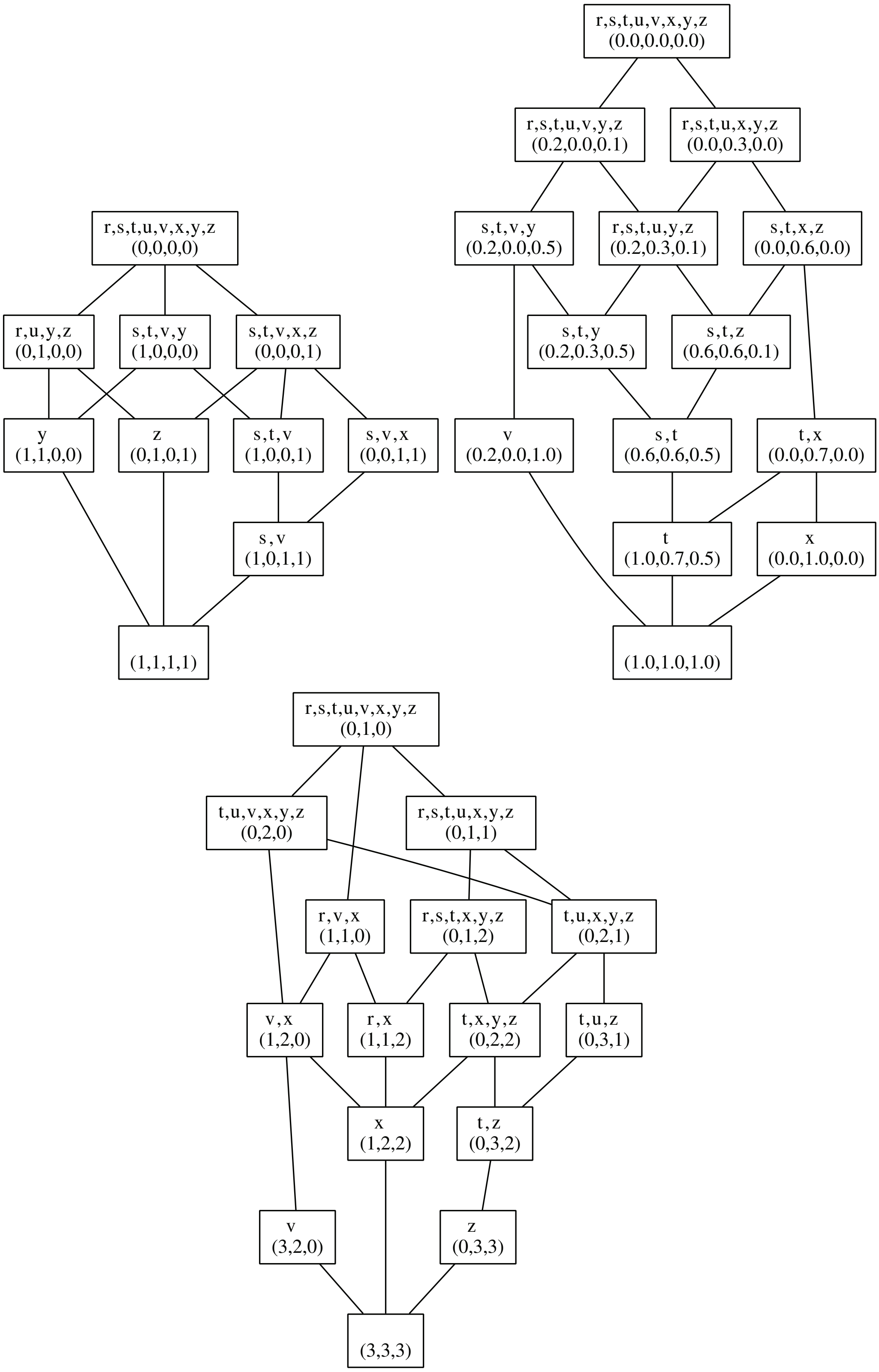

Figure 2: Corresponding concept lattices for particular contexts - up-right: binary attributes, up-left: real attributes, bottom: 4-valued attributes

	$\overline{A}_1$	$\overline{A}_2$	$\overline{A}_3$
r	$\{r,u,y,z\};(0,1,0,0)$	$\{r,s,t,u,y,z\};(0.2,0.3,0.1)$	$\{r,x\};(1,1,2)$
s	$\{s,v\};(1,0,1,1)$	$\{s,t\};(0.6,0.6,0.5)$	$\{r,s,t,x,y,z\};(0,1,2)$
t	$\{s,t,v\};(1,0,0,1)$	$\{t\};(1.0,0.7,0.5)$	$\{t,z\};(0,3,2)$
u	$\{r,u,y,z\};(0,1,0,0)$	$\{r,s,t,u,y,z\};(0.2,0.3,0.1)$	$\{t,u,z\};(0,3,1)$
v	$\{s,v\};(1,0,1,1)$	$\{v\};(0.2,0.0,1.0)$	$\{v\};(3,2,0)$
x	$\{s,v,x\};(0,0,1,1)$	$\{x\};(0.0,1.0,0.0)$	$\{x\};(1,2,2)$
y	$\{y\};(1,1,0,0)$	$\{s,t,y\};(0.2,0.3,0.5)$	$\{t,x,y,z\};(0,2,2)$
z	$\{z\};(0,1,0,1)$	$\{s,t,z\};(0.6,0.6,0.1)$	$\{z\};(0,3,3)$

Table 3: Incidence relation $\ddot{R}$ of new object-attribute model.

Now we will use the obtained one-sided concept lattices as truth value structures for our new object attribute model with three element attribute set. From the former object-attribute model we have $A_1 = \{a_1, a_5, a_6, a_{10}\}$, $A_2 = \{a_2, a_4, a_7\}$ and $A_3 = \{a_3, a_8, a_9\}$, hence $\overline{A}_1$ will denotes new complex attribute with truth value structure equal to the concept lattice on the up-left part of the Figure 2. Similarly $\overline{A}_2$ and $\overline{A}_3$ will denote new complex attributes with truth value structure equal to the concept lattice on the up-right and bottom part of the Figure 2 respectively. The incidence relation $\ddot{R}$ is described in the Table 3.

Let us remark that $\ddot{R}(b, \overline{A}_i) = (\downarrow_i\uparrow_i (b), \uparrow_i (b))$ for all $b \in B$ and $i = 1, 2, 3$. Now we can use the result of Theorem 2 and obtain another interpretation of fuzzy attribute subsets in the basic model. Since generalized one-sided concept lattices defined by (5), (6) and by (16), (17) respectively are isomorphic (more over the sets of extents coincide), we present the one-sided concept lattice given by basic model (see Figure 3).

This lattice is obtained by conventional method and in some cases (especially when the number of of object and attributes rise dramatically) the interpretation of object clusters using attribute fuzzy subsets become problematic or less clear.

The Theorem 2 gives possibility to characterize object clusters by fuzzy subsets of concept lattices. Since concepts in concept lattices are ordered pairs of type (extent,intent) and set of extents uniquely determines concept lattice, we can give the characterization of object subsets by extents (again object subsets) of derived submodels.

As an example consider the concept $\big(\{r,s,t,y,z\};(0,0.2,0,0.3,0,0,0.1,1,2,0)\big)$ from lattice depicted on Figure 3. One possible (traditional interpretation) of this concept is that the set of objects $\{r,s,t,y,z\}$ is determined by generalized fuzzy subset $(0,0.2,0,0.3,0,0,0.1,1,2,0) \in \prod_{i=1}^{10} \mathcal{L}(a_i)$, which represents some kind of threshold for given subset of object.

Another possible interpretation is due to Theorem 2. In this case we can characterize the set of objects $\{r,s,t,y,z\}$ as the ordered triple

$$\big(\downarrow_1\uparrow_1 (\{r,s,t,y,z\}), \downarrow_2\uparrow_2 (\{r,s,t,y,z\}), \downarrow_3\uparrow_3 (\{r,s,t,y,z\})\big)$$

where for all $i = 1, 2, 3$ the subset $\downarrow_i\uparrow_i (\{r,s,t,y,z\})$ can be found in concept lattices corresponding to the derived submodels. In our case these concept lattices are depicted on Figure 2. To be more concrete, the value $\downarrow_i\uparrow_i (X)$ for any subset $X \subseteq B$ can be found as the smallest subset in given concept lattice containing the set $\{r,s,t,y,z\}$. This follows from the fact that the set of extents in any concept lattice forms closure system, hence the smallest subset containing given set is equal to the intersection of all subsets containing it. Particularly for our example we obtain the triple

$$\big(\{r,s,t,u,v,x,y,z\}, \{r,s,t,u,y,z\}, \{r,s,t,x,y,z\}\big).$$

As we can see $\{r,s,t,y,z\} \subsetneq \downarrow_i\uparrow_i (\{r,s,t,y,z\})$ for each $i = 1, 2, 3$, hence information about this cluster of object is not obtained in any of the three submodels of the former object-

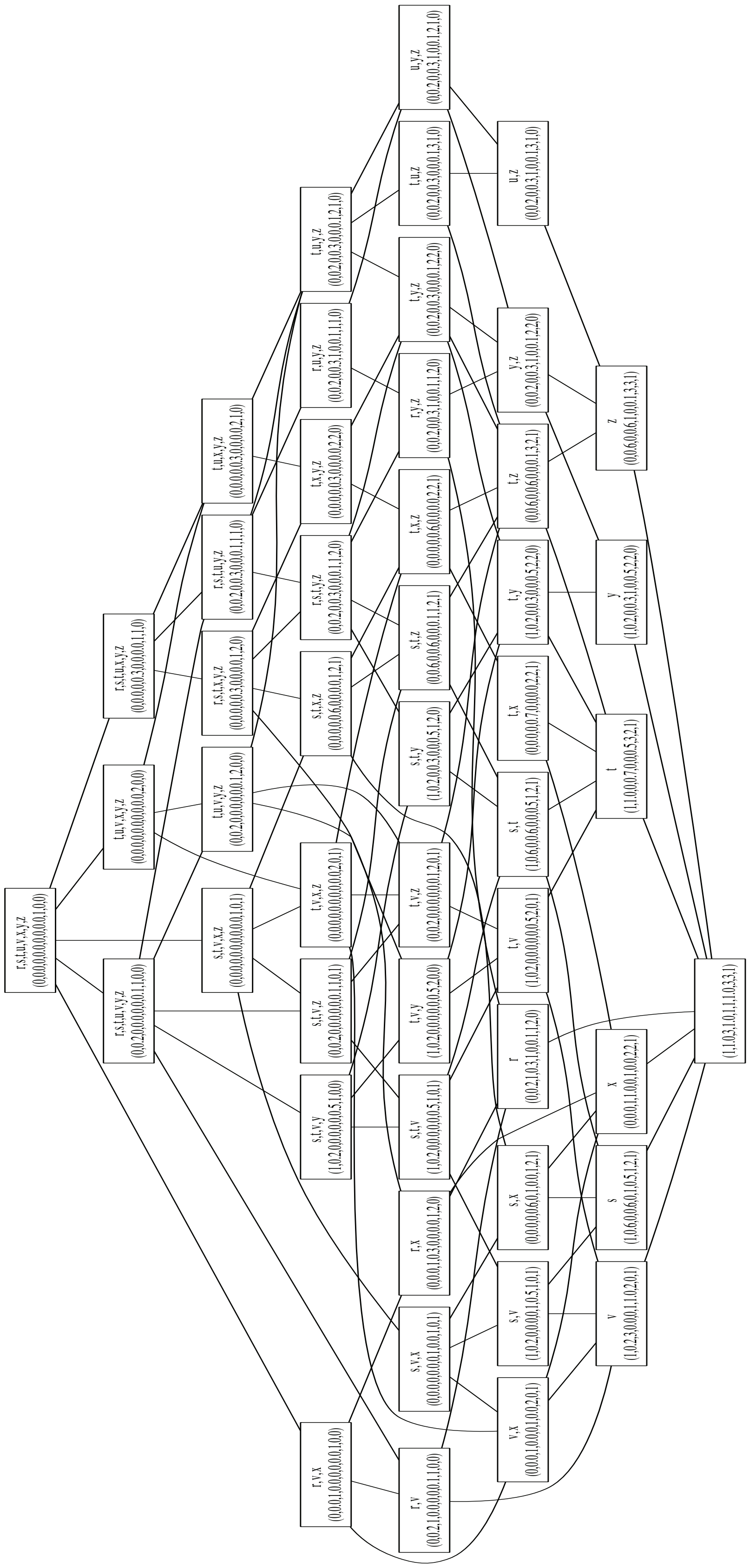

Figure 3: Concept lattice (top element is on the left side, bottom element on the right side).

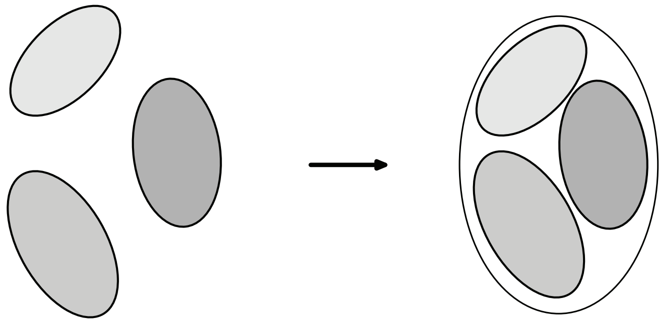

Figure 4: Merging different object-attribute models into one complex model.

attribute model. This kind of information can be useful for analysis of more object-attribute models, which are merged together (see Figure 4 for schematic description and [6] for theoretical explanation).

In this case this method allow us to identify all subsets of objects which are not contained in any previous model but form extents in newly formed complex model. Moreover these extents can be characterized using previous submodels, thus can be interpreted in well-known framework of one-sided concept lattices.

Finally, we also mention that result of Theorem 2 gives reduction of considered object-attribute model in two aspects. First, the number of attributes is smaller. Further, using one-sided concept lattices derived from submodels as truth value structure provide dramatical decrease and simplification of complete lattices tied with fuzzy subsets of attributes with same truth value structures $\mathcal{L}_i^{A_i}$ for all $i \in I$.

Acknowledgments

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