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Sampling Individually Fundamental Simplexes as Sets of Players' Mixed Strategies in Finite Noncooperative Game for Applicable Approximate Nash Equilibrium Situations with Possible Concessions

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Abstract

In finite noncooperative game, a method for finding approximate Nash equilibrium situations is developed. The method is prior-based on sampling fundamental simplexes being the sets of players' mixed strategies. Whereas the sampling is exercised, the sets of players' mixed strategies are mapped into finite lattices. Sampling steps are envisaged dissimilar. Thus, each player within every dimension of its simplex selects and controls one's sampling individually. For preventing approximation low quality, however, sampling steps are restricted. According to the restricted sampling steps, a player acting singly with minimal spacing over its lattice cannot change payoff of any player more than by some predetermined magnitude, being specific for each player. The finite lattice is explicitly built by the represented routine, where the player's mixed strategies are calculated and arranged. The product of all the players' finite lattices approximates the product of continuous fundamental simplexes. This re-defines the finite noncooperative game in its finite mixed extension on the finite lattices' product. In such a finite-mixed-extension-defined game, the set of Nash equilibrium situations may be empty. Therefore, approximate Nash equilibrium situations are defined by the introduced possible payoff concessions. A routine for finding approximate equilibrium situations is represented. Approximate strong Nash equilibria with possible concessions are defined, and a routine for finding them is represented as well. Acceleration of finding approximate equilibria is argued also. Finally, the developed method is discussed to be a basis in stating a universal approach for the finite noncooperative game solution approximation implying unification of the game solvability, applicability, realizability, and adaptability.

Keywords: finite noncooperative game, fundamental simplex, sampling, approximation, approximate Nash equilibrium situations, mapping into a finite set, strong Nash equilibria, payoff concession

1. Noncooperative-game models

In general conception, noncooperative-game modeling is used for allocating resources rationally when they are exceeded with pretensions. Otherwise, if they are not, the question is how to divide resources fairly. Noncooperative-game models are applied in economics [1], [2], ecology [3], [4], technics [5], [6], and social sciences [2], [3]. They allow to model and optimize interaction among economical subjects (enterprises), biological species, discharging queues (servers), schedules, etc. Also uncertainties are removed [7], [8], or reduced [7], [9], [10].

While modeling and practicing solution, finite noncooperative game (FNCG) is preferred to infinite one. The preference is explained with that FNCG is generally solved faster. Besides, FNCG solution in mixed strategies is suitable for practicing it as a mixed strategy in FNCG has finite support [7], [11]. Unlike this, solution of infinite game may contain infinite supports [7], [12]. While practicing such solution, the player whose solution strategy support is infinite selects ever the support pure strategies set of zero measure [12], [13], [14]. Therefore, efficient game modeling implies operations on FNCG.

2. FNCG solution approximation

FNCG is not always solved easily, if in mixed strategies. There is no universal algorithm for finding Nash equilibrium situations in FNCG [7], [12], [15]. Exceptions are [7], [16], [17], [18], [10] bimatrix games (when there are just two players) and dyadic games (every player has just two pure strategies).

For an FNCG solution (in mixed strategies), sometimes it is fast and reasonable to approximate it rather than searching the exact solution. In fact, FNCG solution approximation is either finding Nash equilibrium strategies approximately or rounding probabilities therein [19], [13], [20]. Known polynomial algorithms for approximating Nash equilibria fit bimatrix games [19], but any class of non-dyadic FNCG with three players and more requires specific approach [7], [11], [12], [15]. Rounding probabilities is needed to have one or two digits after decimal point (DADP). This lets approximate statistical frequencies to the rounded probabilities while practicing, else the practiced result is off the equilibrium [13], [14].

As applicability of Nash equilibria in mixed strategies bears on probabilities with minimal number of DADP, FNCG solution approximation might be preferably started with searching through mixed strategies of such probabilities. Quantity of probabilities with finite number of DADP is finite. It implies sampling the sets of players' mixed strategies. In this way, those sets being infinite are mapped into finite ones. With such sets' finite approximation, a universal approach for FNCG solution approximation can be stated.

3. Research goal and tasks

Regarding not only limited applicability of Nash equilibria in mixed strategies, but also analytical and computational difficulties in searching exact solutions of FNCG, a method for finding applicable approximate Nash equilibrium situations must be developed. While accomplishing this, the following tasks are to be fulfilled:

1. State preliminaries on FNCG (notations and indexing).

2. Map fundamental simplexes as sets of players' mixed strategies into finite lattices.

3. Envisage controllable sample step within every dimension of a simplex.

4. For preventing approximation low quality, formulate restrictions on sampling steps.

5. Set out a routine for building lattices approximating the sets of players' mixed strategies.

6. Introducing possible payoff concessions, state a routine for finding approximate Nash equilibrium situations.

7. The similar routine should be stated for strong Nash equilibria with possible concessions.

8. Estimate periods for solving FNCG with approximate equilibria.

9. Argue for acceleration of finding approximate equilibria.

Fulfilling these tasks drives to a universal approach for FNCG solution approximation. And if numbers of DADP and the game cycles are proper, this solution is fully applicable: after having practiced, statistical frequencies approximate enough to support probabilities [13], [14].

4. Preliminaries

Take FNCG

$$\left\langle \left\{ X_n = \left\{ x_{nm} \right\}_{m=1}^{M_n} \right\}_{n=1}^{N}, \left\{ \mathbf{K}_n = \left[k_J^{\langle n \rangle} \right]_{\mathscr{F}} \right\}_{n=1}^{N} \right\rangle \tag{1}$$

of $N \in \mathbb{N} \setminus \{1\}$ players, where X_n is set of pure strategies of the n-th player, and $\mathbf{K}_n$ is its payoff matrix, whose format is

$$\mathscr{F} = \bigtimes_{r=1}^{N} M_r \text{ by } M_r \in \mathbb{N} \setminus \{1\}$$

and indexing

$$J = \{ j_i \}_{i=1}^{N}, \quad j_i \in \left\{ \overline{1, M_i} \right\} \quad \forall\, i = \overline{1, N}. \tag{2}$$

In FNCG (1), the set of all mixed strategies of the n-th player is $(M_n - 1)$-dimensional fundamental simplex

$$\mathcal{P}_n = \left\{ \mathbf{P}_n = \left[p_{nm} \right]_{1 \times M_n} \in \mathbb{R}^{M_n} : p_{nm} \in [0; 1], \sum_{m=1}^{M_n} p_{nm} = 1 \right\}. \tag{3}$$

In situation

$$\left\{\mathbf{P}_i\right\}_{i=1}^{N} \in \bigtimes_{r=1}^{N} \mathcal{P}_r$$

of FNCG (1), the n-th player's expected payoff is

$$v_n\left(\left\{\mathbf{P}_i\right\}_{i=1}^{N}\right) = \sum_{j_q=\overline{1,\, M_q},\; q=\overline{1,\, N}} \left(k_J^{\langle n \rangle} \cdot \prod_{r=1}^{N} p_{rj_r} \right). \tag{4}$$

Now, after these preliminaries, every fundamental simplex (3) is going to be mapped into a finite lattice.

5. Mapping fundamental simplexes as sets of players' mixed strategies into lattices

Mapping an infinite Euclidean finite-dimensional subset (fundamental simplex) into finite one means selecting sequences of points by a rule. The first part of the rule is that all the pure strategies belong to lattice. The second one is that, for keeping the sample step controllable, may every dimension have its own step. Let s_{nm}^{-1} be the sampling step along m-th dimension of simplex (3). Due to the first part of the mapping rule, $s_{nm} \in \mathbb{N} \setminus \{1\}$. But numbers $\left\{s_{nm}\right\}_{m=1}^{M_n}$ must be such that the sum of the n-th player's all selected probabilities be equal to 1. Consequently, one of these numbers doesn't make sense. Thus, without loss of generality, simplex (3) is mapped into the finite lattice

$$\overline{\mathcal{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right) =$$
$$= \left\{ \overline{\mathbf{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right) = \left[\overline{p}_{nm}\left(s_{nm}^{-1}\right)\right]_{1\times M_n} \in \mathcal{P}_n : \right.$$
$$\left. \overline{p}_{nm}\left(s_{nm}^{-1}\right) = \frac{h-1}{s_{nm}},\; h = \overline{1,\, s_{nm}+1} \;\; \forall\, m = \overline{1,\, M_n - 1} \right\} \subset \mathcal{P}_n. \tag{5}$$

Lattice (5) is defined with numbers $\left\{s_{nm}\right\}_{m=1}^{M_n-1}$. Clearly, probability $\overline{p}_{nM_n}\left(s_{nM_n}^{-1}\right)$ in (5) is written formally, being found as

$$\overline{p}_{nM_n}\left(s_{nM_n}^{-1}\right) = 1 - \sum_{m=1}^{M_n-1} \overline{p}_{nm}\left(s_{nm}^{-1}\right). \tag{6}$$

And number s_{nM_n} is non-constant depending on what numbers $\left\{s_{nm}\right\}_{m=1}^{M_n-1}$ are assigned. Then FNCG (1) is defined in its finite mixed extension on the finite lattice

$$\bigtimes_{r=1}^{N} \bar{\mathcal{P}}_r\left(\left\{s_{rm}^{-1}\right\}_{m=1}^{M_r-1}\right). \tag{7}$$

Thus the finite lattice (7) approximates the product $\bigtimes_{r=1}^{N} \mathcal{P}_r$ of fundamental simplexes.

Dissimilar steps may be needed in the three following cases:

1. When $\left\{M_n\right\}_{n=1}^{N}$ are pretty different, but players would wish to run through their lattices similarly (with nearly equal operation speed over the support pure strategies).

2. Among its pure strategies, a player possesses more important strategies and less important strategies.

3. Some players require lesser numbers of DADP, otherwise they will not implement their mixed strategies from an FNCG solution.

The case 1 and case 3 needn't dissimilar steps for the player (over its pure strategies). And the case 2 is just for that kind of dissimilarity. The lesser numbers $\left\{s_{nm}\right\}_{m=1}^{M_n-1}$ provide the n-th player with faster solution implementation. However, faster solution implementation yields FNCG approximation low quality. So, sampling steps shall be restricted for the low quality prevention [21], [22]. Moreover, numbers $\left\{s_{nm}\right\}_{m=1}^{M_n-1}$ are interdependent in order to ensure the sum of the player's all probabilities is equal to 1.

6. Restrictions on sampling steps

The restriction concerns the player's payoffs. They should not vary much as situation changes minimally over nodes of the finite lattice (7). In this way, approximation low quality is prevented. For the q-th player, minimal change of situation

$$\bar{\mathbf{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \in \bar{\mathcal{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \tag{8}$$

is transition from situation (8) to situation

$$\bar{\mathbf{P}}_q^{\langle 0\rangle}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \in \bar{\mathcal{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \tag{9}$$

such that

$$\left\| \overline{\mathbf{P}}_q \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) - \overline{\mathbf{P}}_q^{\langle 0 \rangle} \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \right\| =$$

$$= \min_{\substack{\overline{\mathbf{P}}_q^{*}\left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \in \overline{\mathcal{P}}_q\left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right),\ \overline{\mathbf{P}}_q^{**}\left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \in \overline{\mathcal{P}}_q\left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \\ \overline{\mathbf{P}}_q^{*}\left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \neq \overline{\mathbf{P}}_q^{**}\left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right)}} \left\| \overline{\mathbf{P}}_q^{*} \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) - \overline{\mathbf{P}}_q^{**} \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \right\| =$$

$$= \sigma_q \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \tag{10}$$

by $q = \overline{1, N}$. The norm in (10) is Euclidean one in $\mathbb{R}^{M_q}$. Following this, the n-th player's payoff variation restriction is that

$$\left| v_n \left(\left\{ \overline{\mathbf{P}}_i \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N} \right) - v_n \left(\left\{ \left\{ \overline{\mathbf{P}}_i \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N} \setminus \left\{ \overline{\mathbf{P}}_q \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \right\} \right\} \cup \right.\right.$$

$$\left.\left. \cup \left\{ \overline{\mathbf{P}}_q^{\langle 0 \rangle} \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right) \right\} \right) \right| \leqslant \alpha_n \tag{11}$$

for some $\alpha_n > 0$ by $q = \overline{1, N}$ and $n = \overline{1, N}$ by (10). Hence, integers $\left\{ \left\{ s_{nm} \right\}_{m=1}^{M_n - 1} \right\}_{n=1}^{N}$ defining the sampling steps $\left\{ \left\{ s_{nm}^{-1} \right\}_{m=1}^{M_n - 1} \right\}_{n=1}^{N}$ mustn't be too small or else inequality (11) is violated.

The restriction (11) at distance (10) for (8) and (9) by $q = \overline{1, N}$ implies that as situation changes minimally over nodes of the finite lattice (7), the n-th player's payoff changes no greater than by magnitude α_n. For the q-th player, distance (10) is the minimal spacing over its lattice $\overline{\mathcal{P}}_q \left(\left\{ s_{qm}^{-1} \right\}_{m=1}^{M_q - 1} \right)$. According to the restricted sampling steps, a player acting singly with minimal spacing over its lattice cannot change payoff of any player more than by some predetermined magnitude, being specific for each player. The lattice minimal spacing depends on how the lattice is built based on (5). Below, a routine for building lattices of players' fundamental simplexes is set out.

7. Routine for building lattices

For finite lattice (5), let

$$U_n = \left| \overline{\mathcal{P}}_n \left(\left\{ s_{nm}^{-1} \right\}_{m=1}^{M_n - 1} \right) \right|$$

and index its elements as

$$\overline{\mathcal{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)=\left\{\overline{\mathbf{P}}_n^{\langle u\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)=\left[\overline{p}_{nm}^{\langle u\rangle}\left(s_{nm}^{-1}\right)\right]_{1\times M_n}\right\}_{u=1}^{U_n}. \tag{12}$$

By a convention, the first element in (12) is the n-th player's first pure strategy x_{n1}, i. e.

$$\overline{\mathbf{P}}_n^{\langle 1\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)=\left[\overline{p}_{nm}^{\langle 1\rangle}\left(s_{nm}^{-1}\right)\right]_{1\times M_n}$$
$$\text{by }\ \overline{p}_{n1}^{\langle 1\rangle}\left(s_{n1}^{-1}\right)=1\ \text{ and }\ \overline{p}_{nm}^{\langle 1\rangle}\left(s_{nm}^{-1}\right)=0\quad\forall\ m=\overline{2,\,M_n}\,. \tag{13}$$

The last element in (12) is the n-th player's last pure strategy x_{nM_n},

$$\overline{\mathbf{P}}_n^{\langle U_n\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)=\left[\overline{p}_{nm}^{\langle U_n\rangle}\left(s_{nm}^{-1}\right)\right]_{1\times M_n}\ \text{ by }\ \overline{p}_{nm}^{\langle U_n\rangle}\left(s_{nm}^{-1}\right)=0$$
$$\forall\ m=\overline{1,\,M_n-1}\ \text{ and }\ \overline{p}_{nM_n}^{\langle U_n\rangle}\left(s_{nM_n}^{-1}\right)=1\,. \tag{14}$$

Elements of the set (12) are arranged from (13) right to (14), where M_n-1 nested loops of this arrangement address themselves to inequality

$$\sum_{m=1}^{M_n-1}\overline{p}_{nm}\left(s_{nm}^{-1}\right)\leqslant 1 \tag{15}$$

on the finite lattice (5). Within the core, i. e. inside the $\left(M_n-1\right)$-th loop, the probability

$$\overline{p}_{nM_n}\left(s_{nM_n}^{-1}\right)=1-\sum_{m=1}^{M_n-1}\overline{p}_{nm}\left(s_{nm}^{-1}\right) \tag{16}$$

is calculated if inequality (15) is true. At the start of the routine for building the lattice (12), first M_n-1 zero probabilities are initialized:

$$\overline{p}_{nm}\left(s_{nm}^{-1}\right)=0\quad\forall\ m=\overline{1,\,M_n-1}\,. \tag{17}$$

Inside the t-th loop, M_n-1-t probabilities are initialized to zero before checking the inequality (15):

$$\overline{p}_{nm}\left(s_{nm}^{-1}\right)=0\quad\forall\ m=\overline{t+1,\,M_n-1} \tag{18}$$

by $t=\overline{1,\,M_n-2}$ (Figure 1).

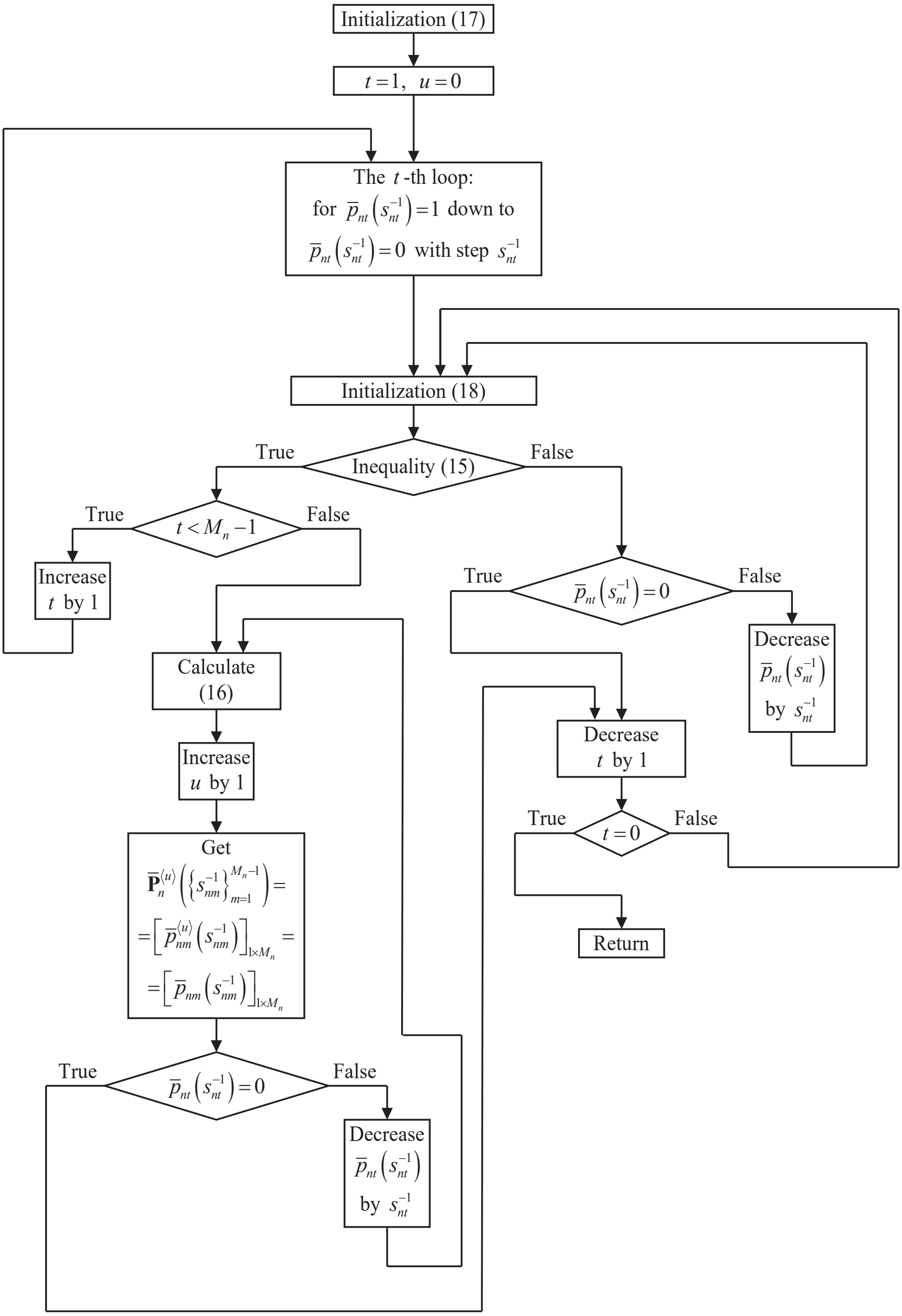


Figure 1. Calculation and arrangement of the set (12) elements

The paradigm of calculating and arranging elements of the set (12) shown in Figure 1 completes the routine for building lattices of players' fundamental simplexes. Once we get in the core loop, the inequality (15) is checked only one time, whereupon $\bar{p}_{nt}\left(s_{nt}^{-1}\right)$ is decreased and the probability (16) is calculated. Once the probability (16) is calculated for $\bar{p}_{nt}\left(s_{nt}^{-1}\right)=0$, being truly $\bar{p}_{n(M_n-1)}\left(s_{n(M_n-1)}^{-1}\right)=0$, the nearest outer loop is addressed by decreasing t by 1. The looped calculation runs until $t=0$.

8. Minimal spacing over lattice

If integers $\left\{s_{nm}\right\}_{m=1}^{M_n-1}$ are identical for the n-th player then the sampling step is constant through dimensions of simplex. Let it be s_n^{-1}. In this case, it is possible to determine the minimal spacing over lattice (12) explicitly. By assigning $s_n \equiv s_{nm}$, according to the routine in Figure 1,

$$\bar{\mathbf{P}}_n^{\langle 2\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)=\bar{\mathbf{P}}_n^{\langle 2\rangle}\left(s_n^{-1}\right)=\left[\bar{p}_{nm}^{\langle 2\rangle}\left(s_n^{-1}\right)\right]_{1\times M_n}$$

$$\text{by } \bar{p}_{n1}^{\langle 2\rangle}\left(s_n^{-1}\right)=1-s_n^{-1} \text{ and } \bar{p}_{n2}^{\langle 2\rangle}\left(s_n^{-1}\right)=s_n^{-1}$$

$$\text{for } \bar{p}_{nm}^{\langle 2\rangle}\left(s_n^{-1}\right)=0 \quad \forall\, m=\overline{3,\, M_n}\,, \tag{19}$$

$$\bar{\mathbf{P}}_n^{\langle U_n-1\rangle}\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)=\bar{\mathbf{P}}_n^{\langle U_n-1\rangle}\left(s_n^{-1}\right)=\left[\bar{p}_{nm}^{\langle U_n-1\rangle}\left(s_n^{-1}\right)\right]_{1\times M_n}$$

$$\text{by } \bar{p}_{nm}^{\langle U_n-1\rangle}\left(s_n^{-1}\right)=0 \quad \forall\, m=\overline{1,\, M_n-2}$$

$$\text{and } \bar{p}_{n(M_n-1)}^{\langle U_n-1\rangle}\left(s_n^{-1}\right)=s_n^{-1} \text{ for } \bar{p}_{nM_n}^{\langle U_n-1\rangle}\left(s_n^{-1}\right)=1-s_n^{-1}\,. \tag{20}$$

As it is easy to see, the n-th player's lattice (12) written here as $\bar{\mathcal{P}}_n\left(s_n^{-1}\right)$ by the sampling step integer s_n becomes fully regular having identical distance between its nodes. This distance is

$$\min_{\substack{\bar{\mathbf{P}}_n^{*}\left(s_n^{-1}\right)\in\bar{\mathcal{P}}_n\left(s_n^{-1}\right),\, \bar{\mathbf{P}}_n^{**}\left(s_n^{-1}\right)\in\bar{\mathcal{P}}_n\left(s_n^{-1}\right)\\ \bar{\mathbf{P}}_n^{*}\left(s_n^{-1}\right)\neq\bar{\mathbf{P}}_n^{**}\left(s_n^{-1}\right)}}\left\|\bar{\mathbf{P}}_n^{*}\left(s_n^{-1}\right)-\bar{\mathbf{P}}_n^{**}\left(s_n^{-1}\right)\right\|=$$

$$=\min_{\overline{l=1,\, U_n-1},\ \overline{u=l+1,\, U_n}}\left\|\bar{\mathbf{P}}_n^{\langle l\rangle}\left(s_n^{-1}\right)-\bar{\mathbf{P}}_n^{\langle u\rangle}\left(s_n^{-1}\right)\right\|=$$

$$=\left\|\bar{\mathbf{P}}_n^{\langle 1\rangle}\left(s_n^{-1}\right)-\bar{\mathbf{P}}_n^{\langle 2\rangle}\left(s_n^{-1}\right)\right\|=\sqrt{s_n^{-2}+s_n^{-2}}=\frac{\sqrt{2}}{s_n}\,. \tag{21}$$

The n-th player cannot change its mixed strategy less than by (21). And if all the players use their own constant steps $\left\{s_n^{-1}\right\}_{n=1}^{N}$, then the n-th player's payoff variation restriction means that if the q-th player changes its strategy by $\frac{\sqrt{2}}{s_q}$ then the n-th player's payoff changes no more than by α_n.

For non-identical integers $\{s_{nm}\}_{m=1}^{M_n-1}$, minimal spacing over the n-th player's lattice (12) is not deduced. In this case, each player has its own lattice minimal spacing. For the n-th player, it is $\sigma_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)$ by denotation (10). This minimal spacing calculated over the player's finite lattice is like its resolution.

9. Approximate Nash equilibrium situations with possible concessions

In FNCG (1), classically defined in its mixed extension on the product $\bigtimes_{r=1}^{N}\mathcal{P}_r$ of continuous fundamental simplexes (3), Nash equilibrium situations $\left\{\mathbf{P}_i^*\right\}_{i=1}^{N}$ satisfying inequalities

$$v_n\left(\left\{\left\{\mathbf{P}_i^*\right\}_{i=1}^{N}\setminus\left\{\mathbf{P}_n^*\right\}\right\}\bigcup\left\{\mathbf{P}_n\right\}\right)\leqslant v_n\left(\left\{\mathbf{P}_i^*\right\}_{i=1}^{N}\right)\quad \forall\ \mathbf{P}_n\in\mathcal{P}_n \text{ and } \forall\ n=\overline{1,\,N}\qquad (22)$$

exist ever. In FNCG (1), defined in its finite mixed extension on the finite lattice (7) which approximates the product $\bigtimes_{r=1}^{N}\mathcal{P}_r$ of continuous fundamental simplexes (3), the set of Nash equilibrium situations $\left\{\overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}$ may be empty, because the corresponding N inequalities

$$v_n\left(\left\{\left\{\overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}\setminus\left\{\overline{\mathbf{P}}_n^*\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)\right\}\right\}\bigcup\left\{\overline{\mathbf{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)\right\}\right)\leqslant$$
$$\leqslant v_n\left(\left\{\overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}\right)$$
$$\forall\ \overline{\mathbf{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right)\in\overline{\mathcal{P}}_n\left(\left\{s_{nm}^{-1}\right\}_{m=1}^{M_n-1}\right) \text{ and } \forall\ n=\overline{1,\,N}\qquad (23)$$

constitute a subset of those ones in (22). Therefore, payoff concessions are needed to get a nonempty set of equilibrium situations $\left\{\overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}$ after (23).

Definition 1. In FNCG (1), the node $\left\{ \overline{\mathbf{P}}_i^* \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N}$ of the finite lattice (7) is called equilibrium situation with concessions $\left\{ \beta_n \right\}_{n=1}^{N}$ if

$$v_n \left(\left\{ \left\{ \overline{\mathbf{P}}_i^* \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N} \setminus \left\{ \overline{\mathbf{P}}_n^* \left(\left\{ s_{nm}^{-1} \right\}_{m=1}^{M_n - 1} \right) \right\} \right\} \cup \left\{ \overline{\mathbf{P}}_n \left(\left\{ s_{nm}^{-1} \right\}_{m=1}^{M_n - 1} \right) \right\} \right) \leqslant$$

$$\leqslant v_n \left(\left\{ \overline{\mathbf{P}}_i^* \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N} \right) + \beta_n$$

$$\forall\ \overline{\mathbf{P}}_n \left(\left\{ s_{nm}^{-1} \right\}_{m=1}^{M_n - 1} \right) \in \overline{\mathcal{P}}_n \left(\left\{ s_{nm}^{-1} \right\}_{m=1}^{M_n - 1} \right) \ \text{and} \ \forall\ n = \overline{1, N} \tag{24}$$

by the n-th player's concession $\beta_n \geqslant 0$.

If $\beta_n = 0 \ \ \forall\ n = \overline{1, N}$ then the node $\left\{ \overline{\mathbf{P}}_i^* \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N}$ by (24), classically, is a Nash equilibrium situation. Henceforward, if $\exists\ n_0 \in \left\{ \overline{1, N} \right\}$ such that $\beta_{n_0} > 0$ then this node by (24) is an approximate Nash equilibrium situation. Let it be called $\left\{ \beta_n \right\}_{n \in B}$-equilibrium situation by

$$B = \left\{ q \in \left\{ \overline{1, N} \right\} : \beta_q > 0 \right\} \tag{25}$$

and permitting also cases when the set (25) is empty.

Primarily, inequalities (24) should be verified for null concessions. If set of Nash equilibrium situations appears empty, concessions are necessary. Another necessity of concession is based on that without concessions we may lose Nash equilibrium solutions existing just on the finite lattice (7) as a result of arithmetic, having finite digit precision and roundoff errors. Say, if we have a sampling step $1/3$ then even if equilibrium strategy probabilities are only $1/3$ and $2/3$ we need

$$\beta_n \in \left\{ 10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}, 10^{-10} \right\}$$

or something like that.

For convenience of sweep, the inequalities (24) are stated in the view:

$$v_n \left(\left\{ \left\{ \overline{\mathbf{P}}_i^* \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N} \setminus \left\{ \overline{\mathbf{P}}_n^* \left(\left\{ s_{nm}^{-1} \right\}_{m=1}^{M_n - 1} \right) \right\} \right\} \cup \left\{ \overline{\mathbf{P}}_n^{\langle u_n \rangle} \left(\left\{ s_{nm}^{-1} \right\}_{m=1}^{M_n - 1} \right) \right\} \right) \leqslant$$

$$\leqslant v_n \left(\left\{ \overline{\mathbf{P}}_i^* \left(\left\{ s_{im}^{-1} \right\}_{m=1}^{M_i - 1} \right) \right\}_{i=1}^{N} \right) + \beta_n \quad \forall\ u_n = \overline{1, U_n} \ \text{and} \ \forall\ n = \overline{1, N}. \tag{26}$$

Based on (26), $\left\{\beta_n\right\}_{n \in B}$ -equilibrium situations can be found by the straight search, similarly to searching equilibrium situations in pure strategies in FNCG (1). Surely, strong equilibrium requires to be conceded likewise and even more.

Definition 2. In FNCG (1), the node $\left\{\overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^N$ of the finite lattice (7) is called strong equilibrium situation with concessions $\left\{\gamma_C\right\}_{C \subset \left\{\overline{1,\, N}\right\}}$ for coalitions C if

$$\sum_{n \in C} v_n \left(\left\{ \left\{ \overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^N \setminus \left\{ \overline{\mathbf{P}}_q^*\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\}_{q \in C} \right\} \cup \left\{ \overline{\mathbf{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\}_{q \in C} \right) \leqslant$$
$$\leqslant \sum_{n \in C} v_n \left(\left\{ \overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^N \right) + \gamma_C$$
$$\forall\ \overline{\mathbf{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right) \in \overline{\mathcal{P}}_q\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)$$
$$\text{for } q \in C \text{ and } \forall\ C \subset \left\{\overline{1,\, N}\right\} \tag{27}$$

by concessions $\gamma_C \geqslant 0$.

For convenience of sweep, the inequalities (27) are stated in the view for the straight search:

$$\sum_{n \in C} v_n \left(\left\{ \left\{ \overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^N \setminus \left\{ \overline{\mathbf{P}}_q^*\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\}_{q \in C} \right\} \cup \left\{ \overline{\mathbf{P}}_q^{\langle u_q \rangle}\left(\left\{s_{qm}^{-1}\right\}_{m=1}^{M_q-1}\right)\right\}_{q \in C} \right) \leqslant$$
$$\leqslant \sum_{n \in C} v_n \left(\left\{ \overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^N \right) + \gamma_C$$
$$\forall\ u_q = \overline{1,\, U_q} \text{ for } q \in C \text{ and } \forall\ C \subset \left\{\overline{1,\, N}\right\}. \tag{28}$$

Again, if $\gamma_C = 0\ \ \forall\ C \subset \left\{\overline{1,\, N}\right\}$ then the node $\left\{\overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^N$ by (27), classically, is a strong Nash equilibrium situation. Henceforward, if $\exists\ C_0 \in \left\{\overline{1,\, N}\right\}$ such that $\gamma_{C_0} > 0$ then this node by (27) is an approximate strong Nash equilibrium situation. Let it be called strong $\left\{\gamma_C\right\}_{C \subset \left\{\overline{1,\, N}\right\}}$-equilibrium situation. Consequently, the Nash equilibrium situation is the particular case of $\left\{\beta_n\right\}_{n \in B}$ -equilibrium situation, and the (classical) strong Nash equilibrium situation is the particular case of the strong $\left\{\gamma_C\right\}_{C \subset \left\{\overline{1,\, N}\right\}}$ -equilibrium situation.

10. Routine for finding approximate Nash equilibrium situations

In searching $\left\{\beta_n\right\}_{n\in B}$-equilibrium situations straightforwardly (starting without priorities), every situation

$$\left\{\overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N} = \left\{\overline{\mathbf{P}}_i^{\left\langle u_i^*\right\rangle}\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N} \tag{29}$$

is held at some set $\left\{u_i^* \in \left\{\overline{1, U_i}\right\}\right\}_{i=1}^{N}$ and N payoffs

$$v_n\left(\left\{\overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}\right) + \beta_n \quad \forall\ n = \overline{1, N} \tag{30}$$

are calculated. Launching the routine for the first player, the n-th player's payoff in left side of (26) is calculated, starting from $u_n = 1$. If at some u_n the inequality in (26) fails then next situation (29) is held (Figure 2). If it is true, the n-th player's counter c_n is increased by 1. And if $c_n = U_n \quad \forall\ n = \overline{1, N}$ then a $\left\{\beta_n\right\}_{n\in B}$-equilibrium situation is found, and the counter a for $\left\{\beta_n\right\}_{n\in B}$-equilibrium situations (approximate Nash equilibrium situations) is increased by 1.

Routine for finding strong $\left\{\gamma_C\right\}_{C\subset\left\{\overline{1, N}\right\}}$-equilibrium situations starts identically: every situation (29) is held at some set $\left\{u_i^* \in \left\{\overline{1, U_i}\right\}\right\}_{i=1}^{N}$ and the payoff

$$\sum_{n\in C} v_n\left(\left\{\overline{\mathbf{P}}_i^*\left(\left\{s_{im}^{-1}\right\}_{m=1}^{M_i-1}\right)\right\}_{i=1}^{N}\right) + \gamma_C \tag{31}$$

for the Q_C-th coalition C is calculated, where $Q_C = \overline{1, d_N\left(C\right)}$ and $d_N\left(C\right)$ is total number of coalitions C by the given cardinal $\left|C\right|$ for N players. The routine is launched for the simplest coalitions having $\left|C\right| = 1$. Clearly, these ones are players themselves. Before calculating the Q_C-th coalition in left side of (28), $\left|C\right|$ loops are initialized, where the r-th loop has variable u_{q_r} by $C = \left\{q_r\right\}_{r=1}^{\left|C\right|}$ (Figure 3). If at some C and u_q by $q = q_r$ the inequality (28) fails then next situation (29) is held. If it is true, the coalition counter $c\left(Q_C\right)$ is increased by 1. And if

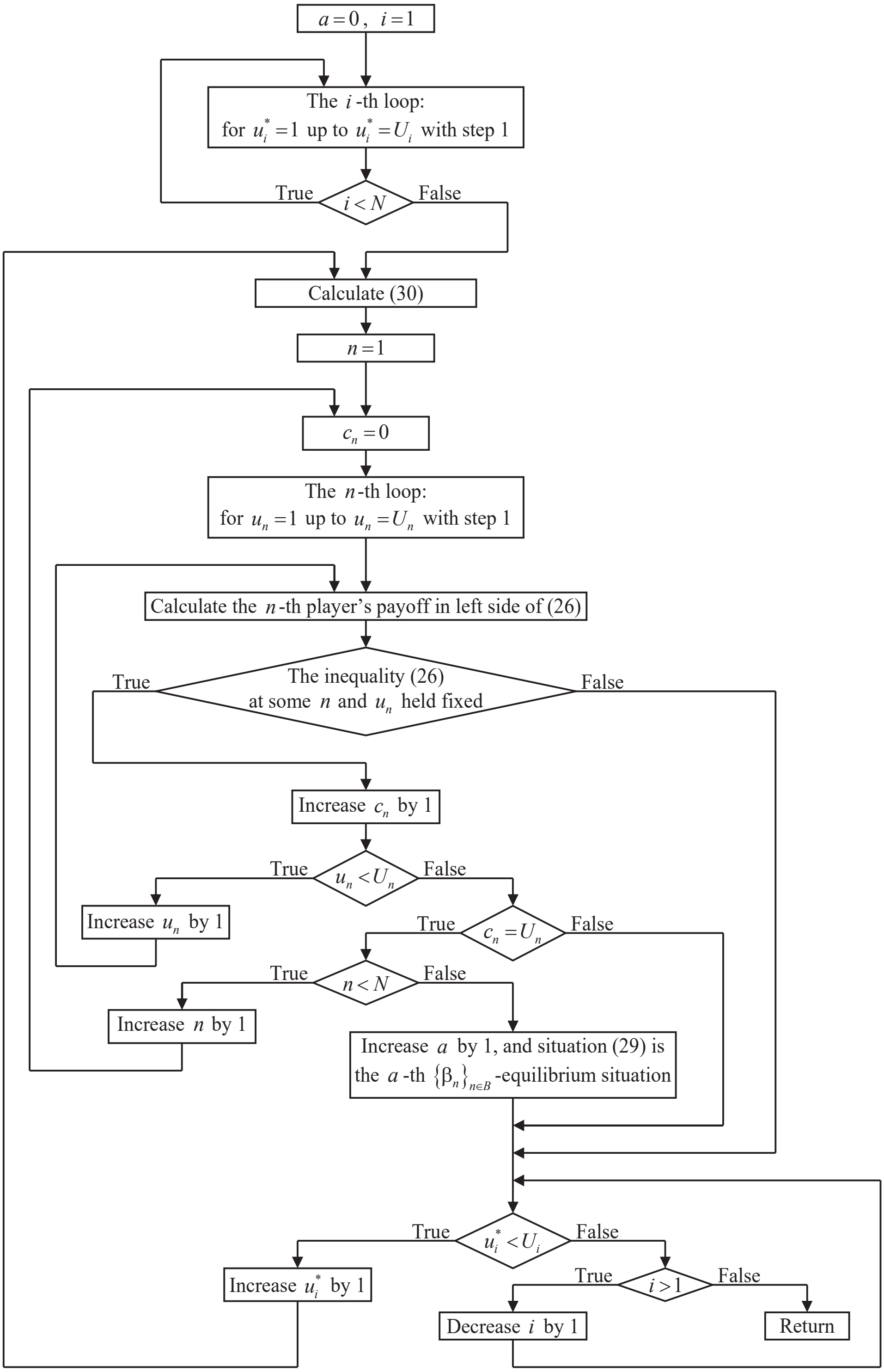


Figure 2. Routine for finding $\{\beta_n\}_{n\in B}$-equilibrium situations

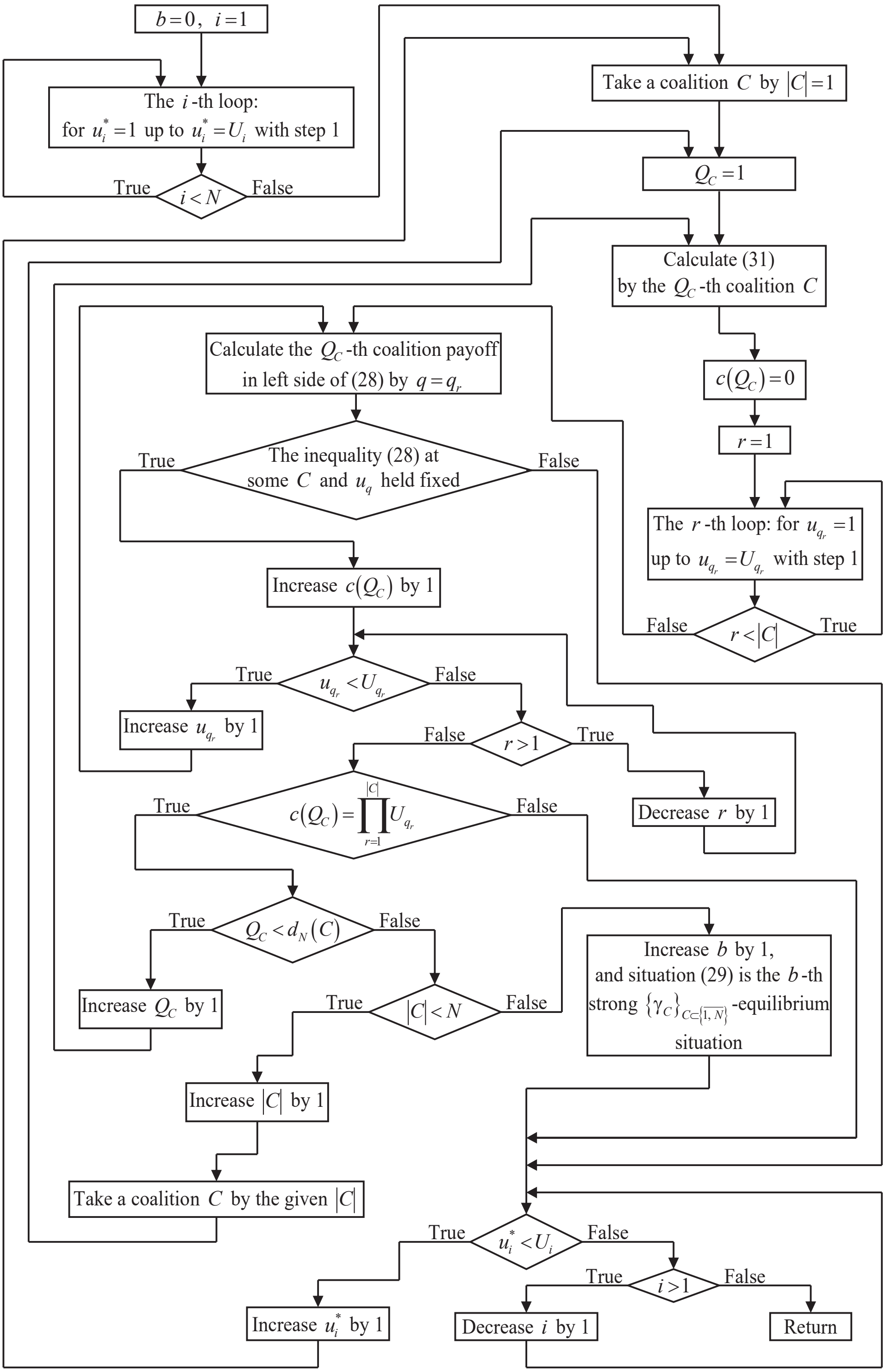


Figure 3. Routine for finding strong $\{\gamma_C\}_{C \subset \{\overline{1, N}\}}$-equilibrium situations

$$c(Q_C) = \prod_{r=1}^{|C|} U_{q_r} \tag{32}$$

then the next coalition is taken without increasing $|C|$ by $Q_C < d_N(C)$. If by the given $|C|$ all coalitions have been taken and run through, $|C|$ is increased by 1 by $|C| < N$ and for the new $|C|$ all corresponding coalitions are taken. If (32) is true $\forall\, Q_C = \overline{1, d_N(C)}$ for all coalitions whose cardinality $|C| = \overline{1, N}$ successively, then a strong $\{\gamma_C\}_{C\subset\{\overline{1, N}\}}$-equilibrium situation is found, and the counter b for strong $\{\gamma_C\}_{C\subset\{\overline{1, N}\}}$-equilibrium situations (approximate strong Nash equilibrium situations) is increased by 1.

The routine for finding $\{\beta_n\}_{n\in B}$-equilibrium situations in Figure 2 is the starter subroutine for finding strong $\{\gamma_C\}_{C\subset\{\overline{1, N}\}}$-equilibrium situations, when N pseudo-coalitions by $|C| = 1$ are taken. That is why searching approximate strong equilibria should be launched directly anyway.

11. Estimation of periods for solving FNCG with approximate equilibria

If concessions are null, likelihood of a loop in Figure 2 or Figure 3 is going to be broken is higher. Consequently, solving FNCG with approximate equilibria is longer. To estimate periods for solving, magnitudes $\{\alpha_n\}_{n=1}^{N}$ and concessions $\{\beta_n\}_{n\in B}$ or $\{\gamma_C\}_{C\subset\{\overline{1, N}\}}$ are adjusted considering N scatters

$$\max_{J=\{j_i\}_{i=1}^{N},\, j_i=\overline{1, M_i},\, i=\overline{1, N}} \left(k_J^{\langle n\rangle}\right) - \min_{J=\{j_i\}_{i=1}^{N},\, j_i=\overline{1, M_i},\, i=\overline{1, N}} \left(k_J^{\langle n\rangle}\right) \text{ by } n = \overline{1, N} \tag{33}$$

of the players' payoffs. For convenience of estimation, it is better to do on normalized payoffs. Thus, for payoff matrices

$$\left\{ \mathbf{G}_n = \left[g_J^{\langle n\rangle} \right]_{\mathscr{F}} \right\}_{n=1}^{N}$$

by indexing (2), affinely equivalent transfer (AET) to FNCG (1) is exercised:

$$\overline{g}_J^{\langle n\rangle} = g_J^{\langle n\rangle} - \min_{I=\{t_i\}_{i=1}^{N},\, t_i=\overline{1, M_i},\, i=\overline{1, N}} \left(g_I^{\langle n\rangle}\right) + \mu_n \text{ by } \mu_n > 0 \tag{34}$$

and

$$k_J^{\langle n \rangle} = \frac{\overline{g}_J^{\langle n \rangle}}{\max\limits_{I=\{t_i\}_{i=1}^N,\ t_i=\overline{1,\ M_i},\ i=\overline{1,\ N}} \left(\overline{g}_I^{\langle n \rangle} \right)} \tag{35}$$

for every player $n = \overline{1,\ N}$. Usually,

$$\mu_n = \mu \quad \forall\ n = \overline{1,\ N} \tag{36}$$

and $\mu = 1$. After the transfer, every player has its payoff equal to 1:

$$k_J^{\langle n \rangle} \in (0;\ 1] \quad \forall\ n = \overline{1,\ N}. \tag{37}$$

If the players' payoffs $\{\mathbf{G}_n\}_{n=1}^N$ are primordially given in the same measuring system, the homogeneous AET to FNCG (1) can be exercised instead of (34) and (35):

$$\overline{g}_J^{\langle n \rangle} = g_J^{\langle n \rangle} - \min_{r=\overline{1,\ N}} \left(\min_{I=\{t_i\}_{i=1}^N,\ t_i=\overline{1,\ M_i},\ i=\overline{1,\ N}} \left(g_I^{\langle r \rangle} \right) \right) + \mu_n \ \text{ by } \ \mu_n > 0 \tag{38}$$

and

$$k_J^{\langle n \rangle} = \frac{\overline{g}_J^{\langle n \rangle}}{\max\limits_{r=\overline{1,\ N}} \left(\max\limits_{I=\{t_i\}_{i=1}^N,\ t_i=\overline{1,\ M_i},\ i=\overline{1,\ N}} \left(\overline{g}_I^{\langle r \rangle} \right) \right)} \tag{39}$$

for every player $n = \overline{1,\ N}$. The assignment (36) is used as well. Homogeneous AET by (38) and (39) leaves just a single player or a few players (generally speaking, not all players) having the maximal payoff equal to 1.

The normalization allows taking

$$\beta_n = \beta \quad \forall\ n = \overline{1,\ N} \tag{40}$$

by β equal to a few hundredths at most. Similarly to (40), γ_C is invariable for the same $|C|$. However, $\gamma_{C_1} \geqslant \gamma_{C_2}$ is recommended for $|C_1| > |C_2|$.

The players' payoffs $\{\mathbf{G}_n\}_{n=1}^N$ will be randomized. So, $g_J^{\langle n \rangle}$ is a value of the standard normal variate. The sampling steps will be identical for simplicity in exemplification and estimation. Having two pure strategies and two players is trivial and, furthermore, bimatrix games are solved exactly [16], [17]. Dyadic games are harder in their solving. So, three players and more will operate their two and three and four pure strategies. For estimation, the processor Intel® Core™ i3-4150 CPU@3.50GHz by 4 GB of RAM is used on 64-bit Windows 7.

Figures 4 — 6 corresponding to $N \in \{3, 4, 5\}$ show periods in seconds taken for finding $\{\beta\}_{n \in \{\overline{1, N}\}}$ -equilibrium situations by

$$\{\beta = 0.01 + 0.002w\}_{w=0}^{20} \quad \text{and} \quad s_{nm} \in \left\{\overline{5, 10}\right\} \quad \forall\; n = \overline{1, N} \quad \text{and} \quad \forall\; m = \overline{1, 2}\,.$$

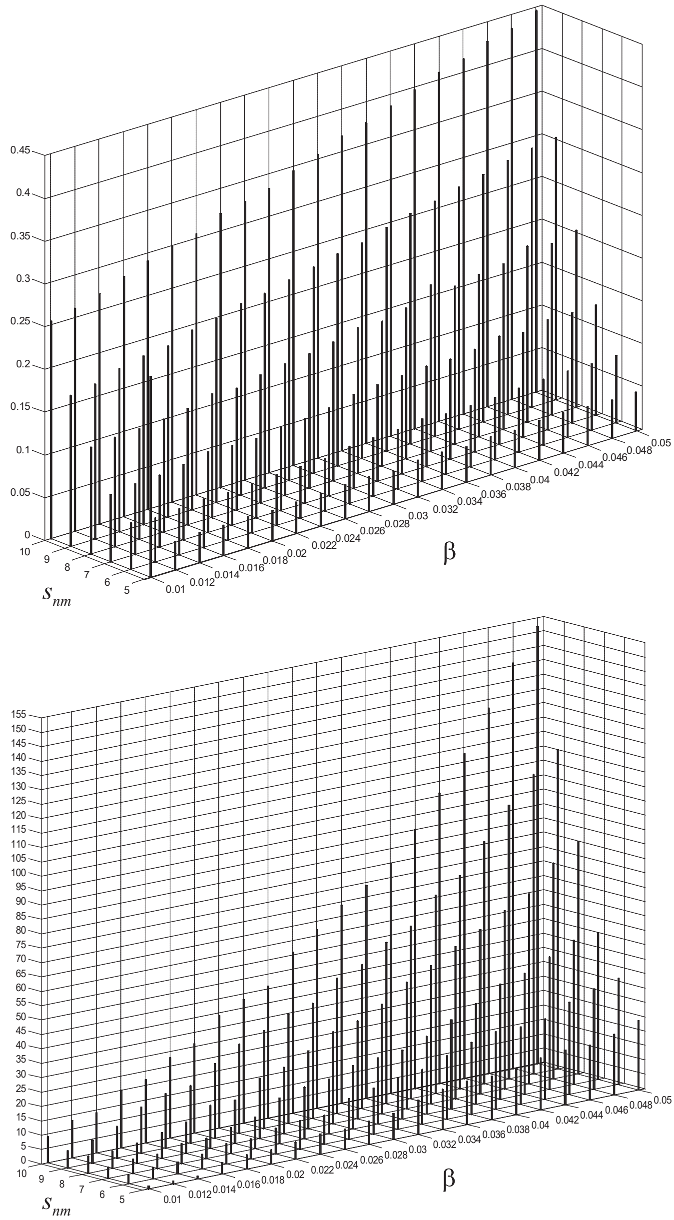


Figure 4. Time taken for finding $\{\beta\}_{n \in \{\overline{1, 3}\}}$ -equilibrium situations in a $2 \times 2 \times 2$ -FNCG and their cardinalities (the lower bar chart)

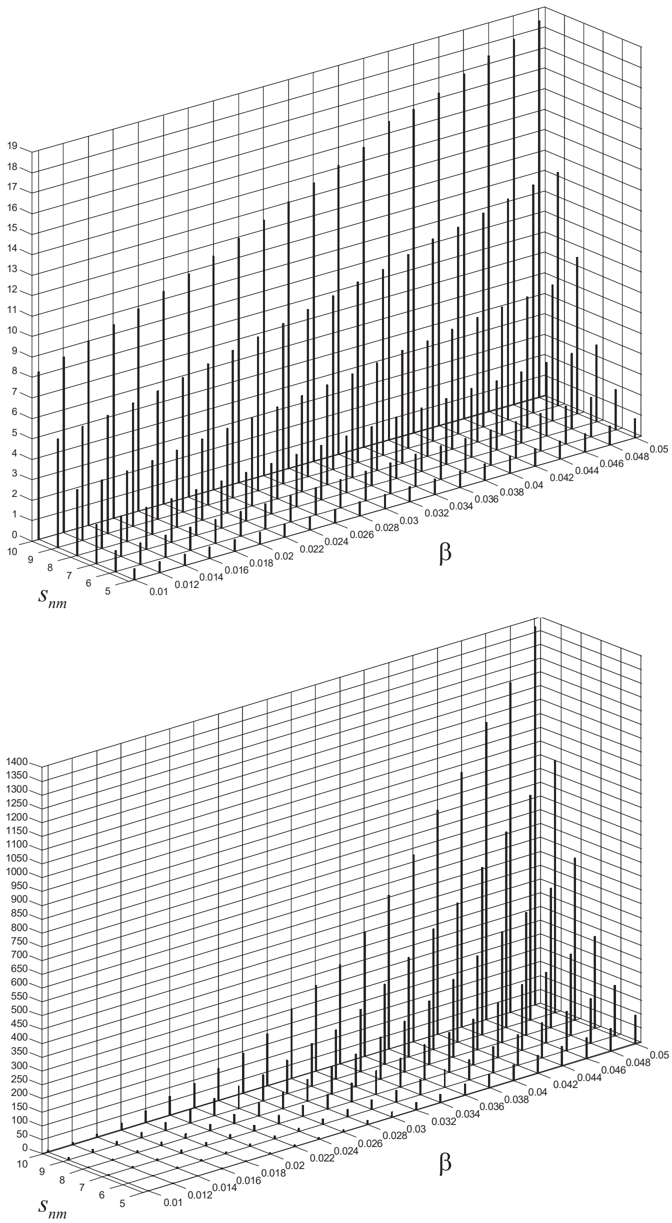
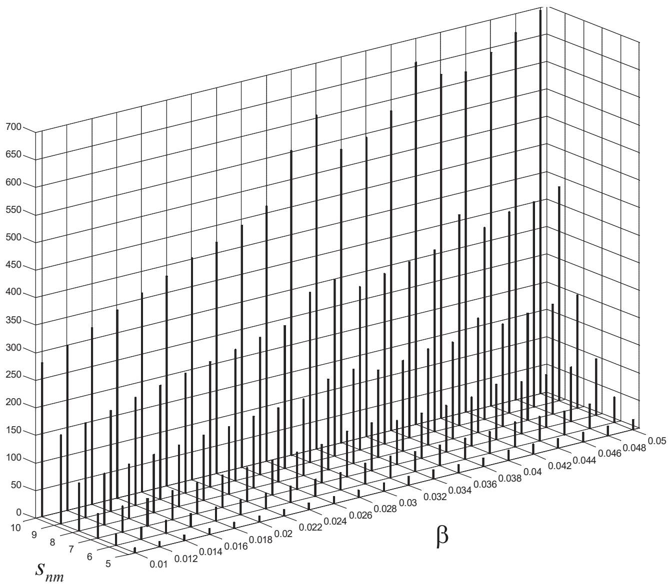


Figure 5. Time taken for finding $\{\beta\}_{n\in\{\overline{1,\,4}\}}$ -equilibrium situations in a $2\times2\times2\times2$ -FNCG and their cardinalities (the lower bar chart)

Cardinalities of sets of those situations are unacceptably great. Consequently, concessions ought to be assigned smaller. Then we obtain a few $\{\beta\}_{n\in\{\overline{1,\,N}\}}$ -equilibrium situations, and it takes far less time, what is exampled in Figures 7 — 10 corresponding to $N \in \{3, 4, 5, 6\}$ and showing periods in seconds taken for finding $\{\beta\}_{n\in\{\overline{1, N}\}}$ -equilibrium situations by

$$\{\beta = 0.001w\}_{w=1}^{5} \text{ and } s_{nm} \in \left\{\overline{5, 10}\right\} \quad \forall\, n = \overline{1, N} \text{ and } \forall\, m = \overline{1, 2}\,.$$



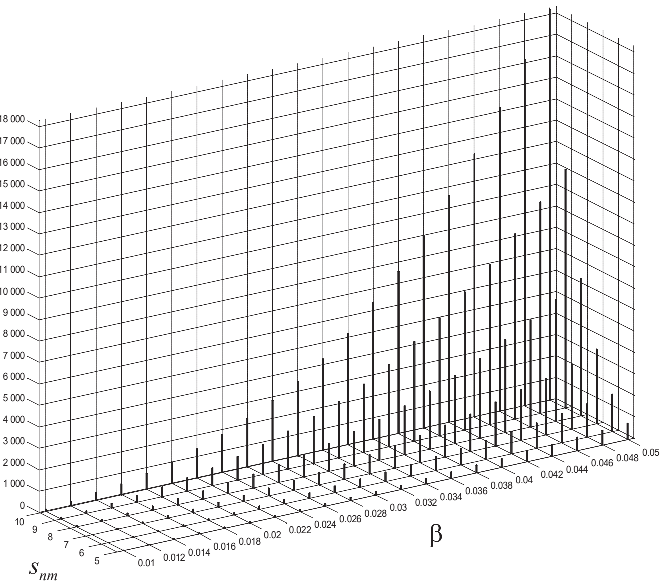


Figure 6. Time taken for finding $\{\beta\}_{n\in\{\overline{1, 5}\}}$ -equilibrium situations in a $\bigtimes_{r=1}^{5} 2$ -FNCG and their cardinalities (the lower bar chart)

Here concessions are decreased in 10 times, and the periods are shortened in about a few times (twice, at least).

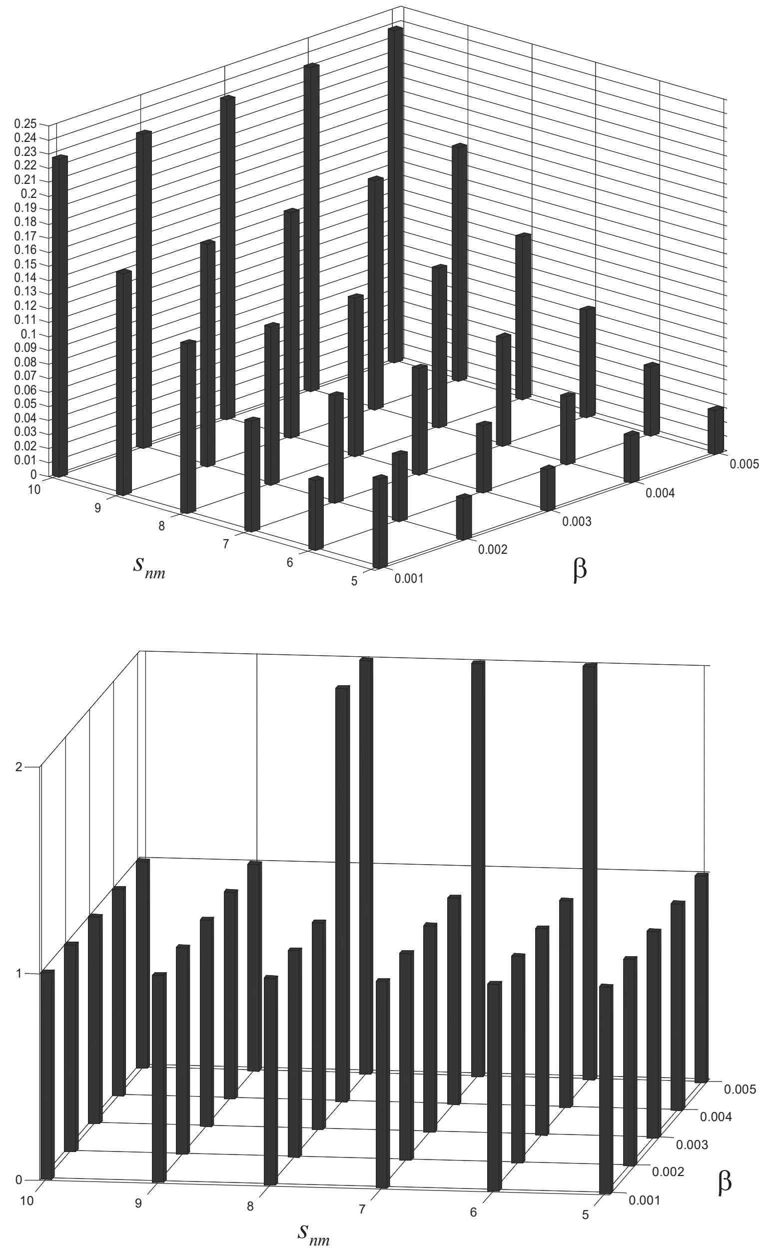


Figure 7. The shortened time periods taken for finding $\{\beta\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations by $\{\beta = 0.001w\}_{w=1}^{5}$ in a $2\times 2\times 2$ -FNCG and their cardinalities (the lower bar chart)

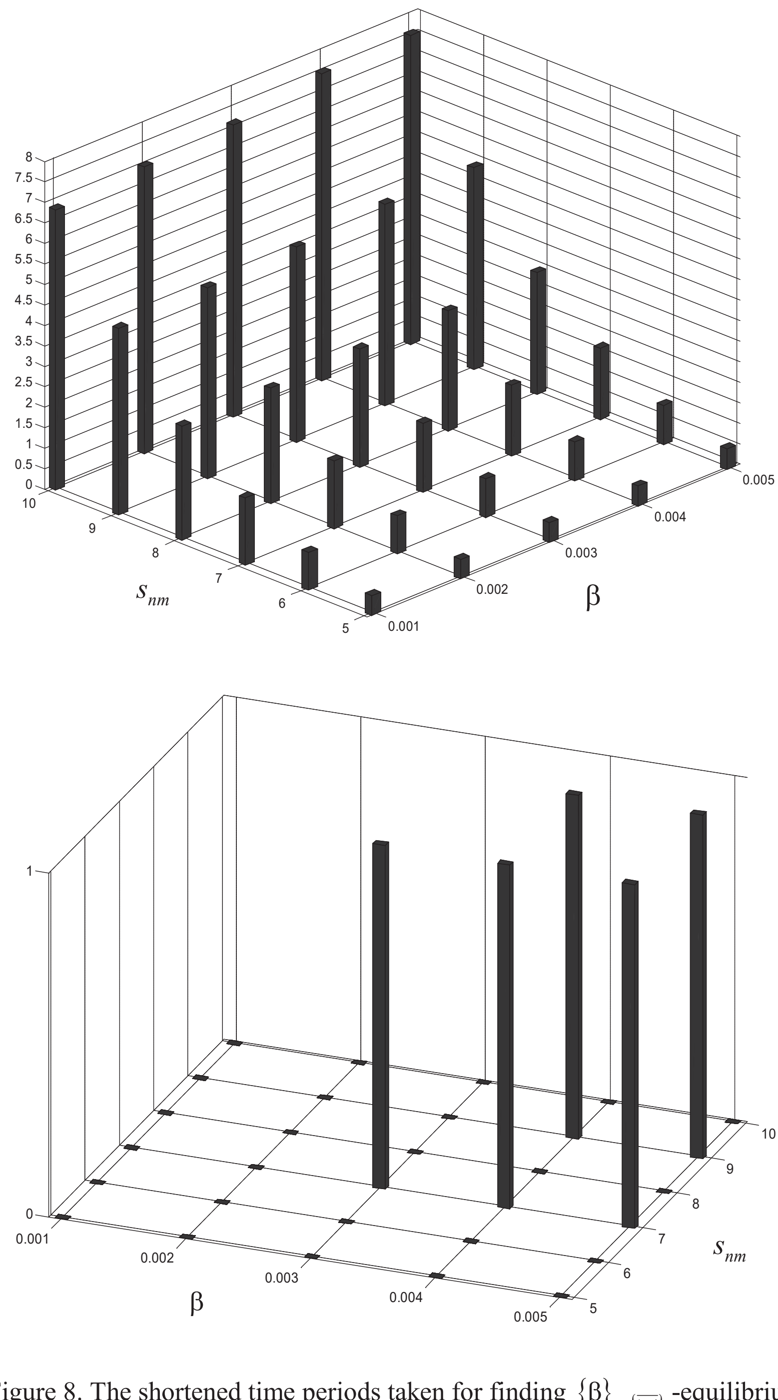


Figure 8. The shortened time periods taken for finding $\{\beta\}_{n\in\{\overline{1,4}\}}$-equilibrium situations by $\{\beta = 0.001w\}_{w=1}^{5}$ in a $2\times 2\times 2\times 2$-FNCG and their cardinalities (the lower bar chart)

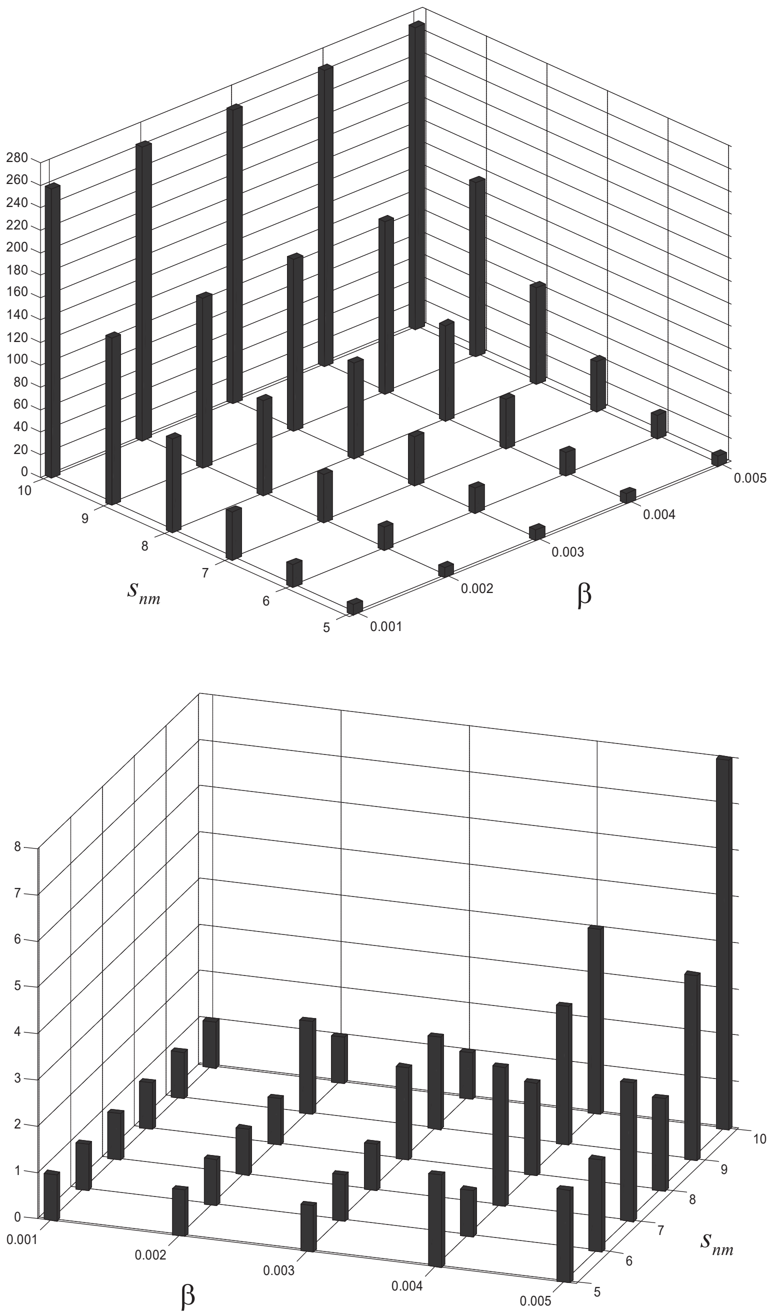
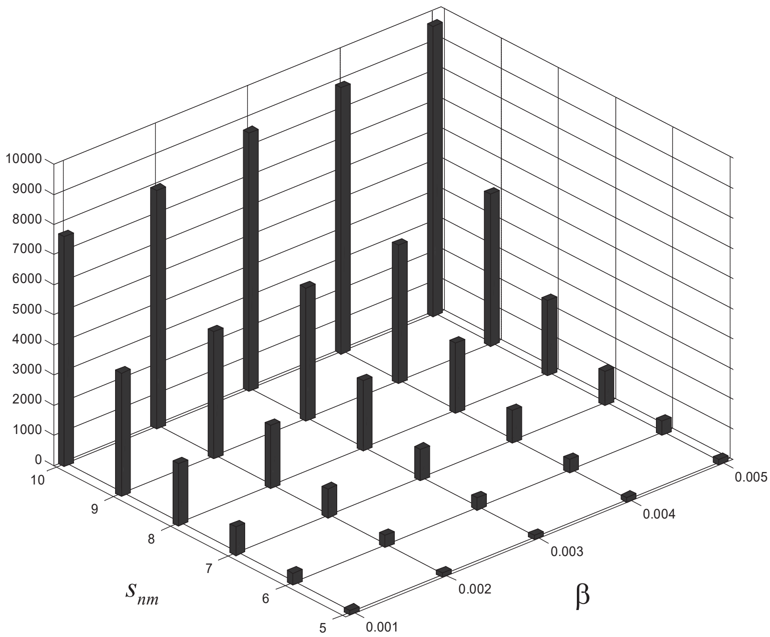


Figure 9. The shortened time periods taken for finding $\{\beta\}_{n\in\{\overline{1,5}\}}$-equilibrium situations by $\{\beta = 0.001w\}_{w=1}^{5}$ in a $\bigtimes_{r=1}^{5} 2$-FNCG and their cardinalities (the lower bar chart)



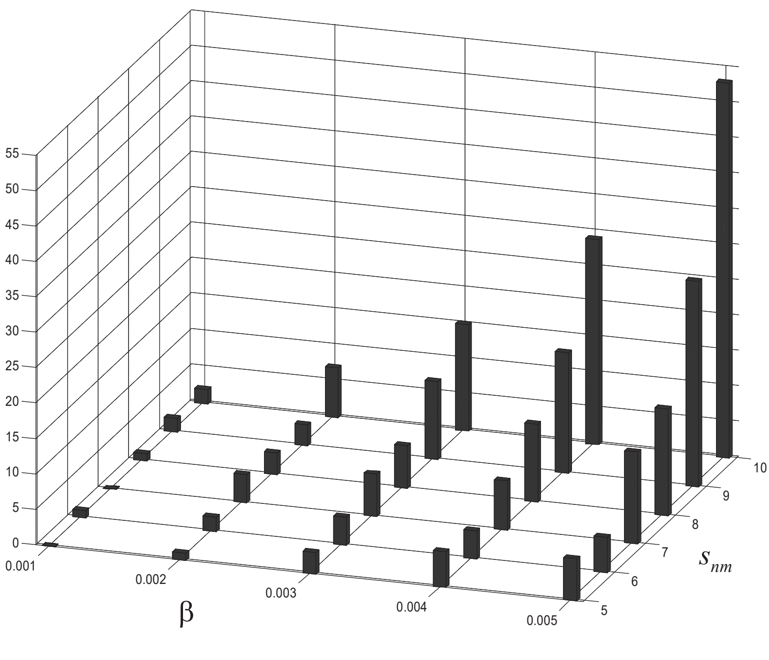


Figure 10. The shortened time periods taken for finding $\{\beta\}_{n\in\{\overline{1,6}\}}$ -equilibrium situations by $\{\beta = 0.001w\}_{w=1}^{5}$ in a $\bigtimes_{r=1}^{6} 2$ -FNCG and their cardinalities (the lower bar chart)

Bad scatters of the periods and the cardinalities are obvious for three players having three pure strategies (Figures 11 — 16). Apparently, decreasing the

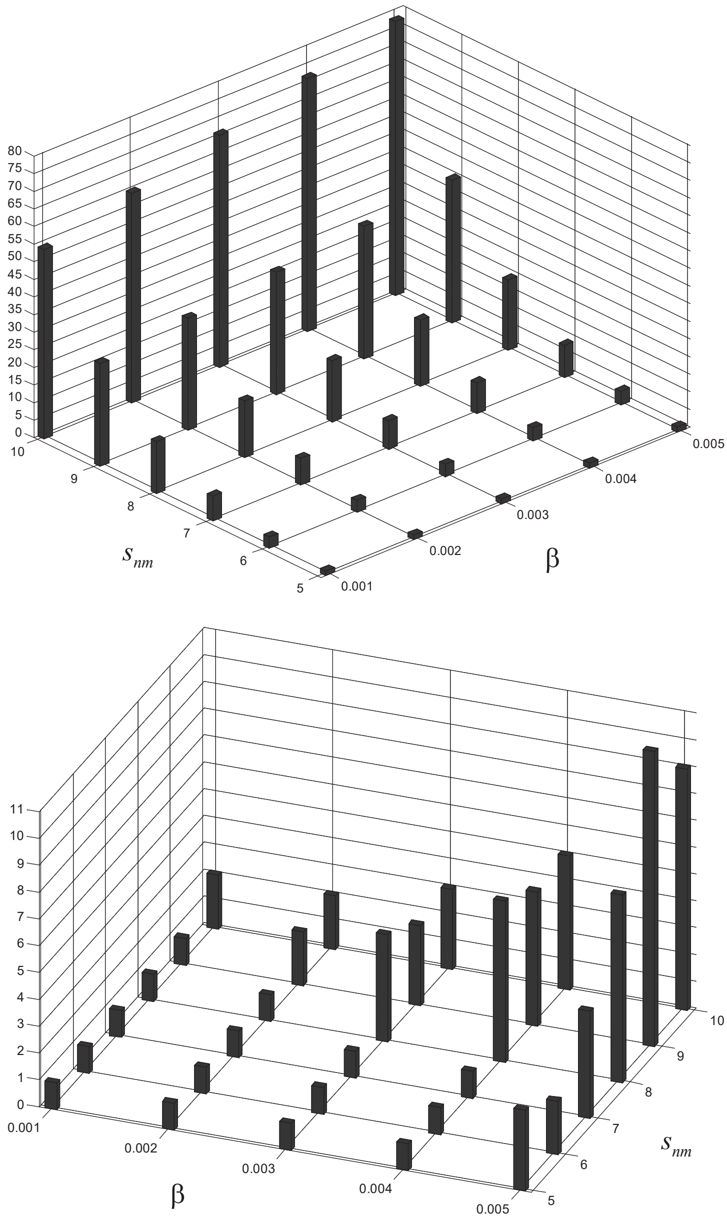
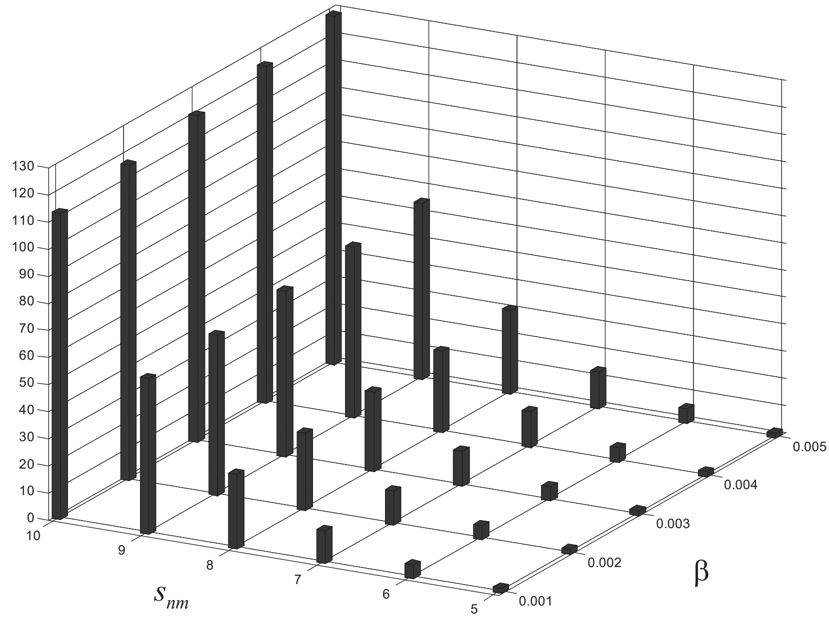


Figure 11. The shortest time periods (in seconds) taken for finding $\{\beta\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations in a $3\times 3\times 3$ -FNCG and their cardinalities (the lower bar chart) by the single $\{0.001\}_{n\in\{\overline{1,3}\}}$ -equilibrium situation for $s_{nm}\in\{5, 6, 7, 8, 9\}$

and by the single $\{0.002\}_{n\in\{\overline{1,3}\}}$ -equilibrium situation for $s_{nm} \in \{5, 6, 7, 8\}$



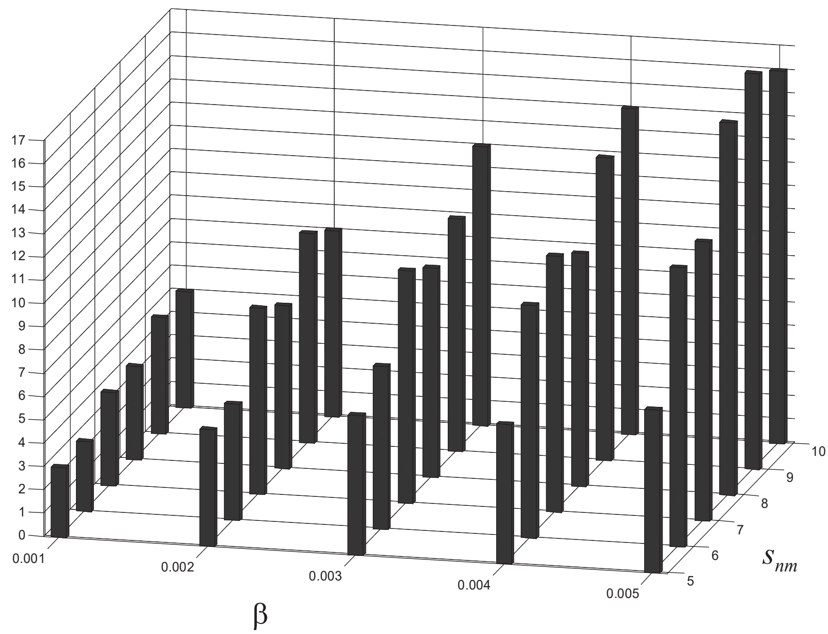
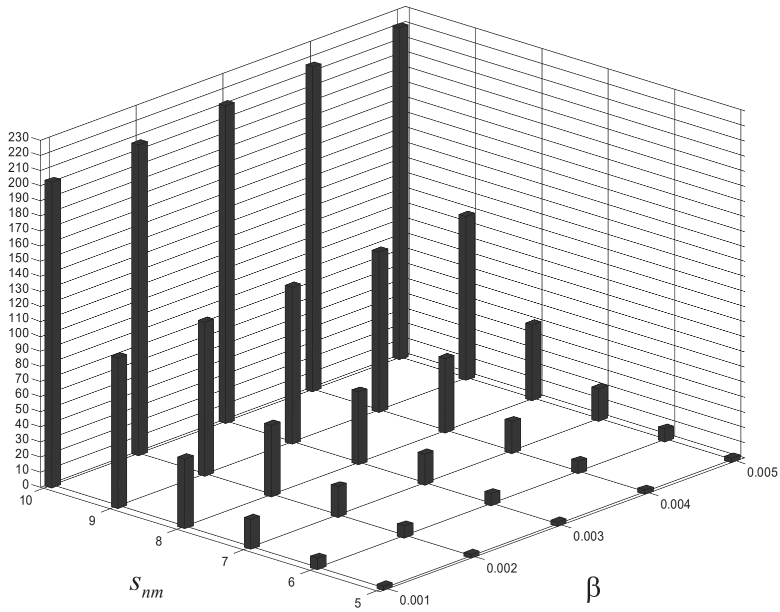


Figure 12. Time periods (in seconds) taken for finding $\{\beta\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations in a $3\times3\times3$ -FNCG are somewhat longer

and their cardinalities (the lower bar chart) are very great



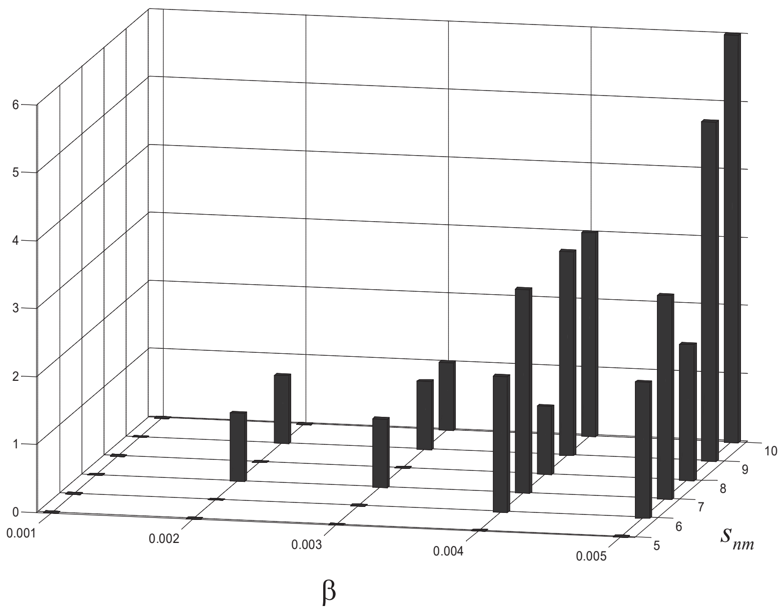


Figure 13. Long time periods (in seconds) taken for finding $\{\beta\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations in a $3\times3\times3$ -FNCG and their cardinalities (the lower bar chart)

by no one $\{0.001\}_{n\in\{\overline{1,3}\}}$ -equilibrium situation

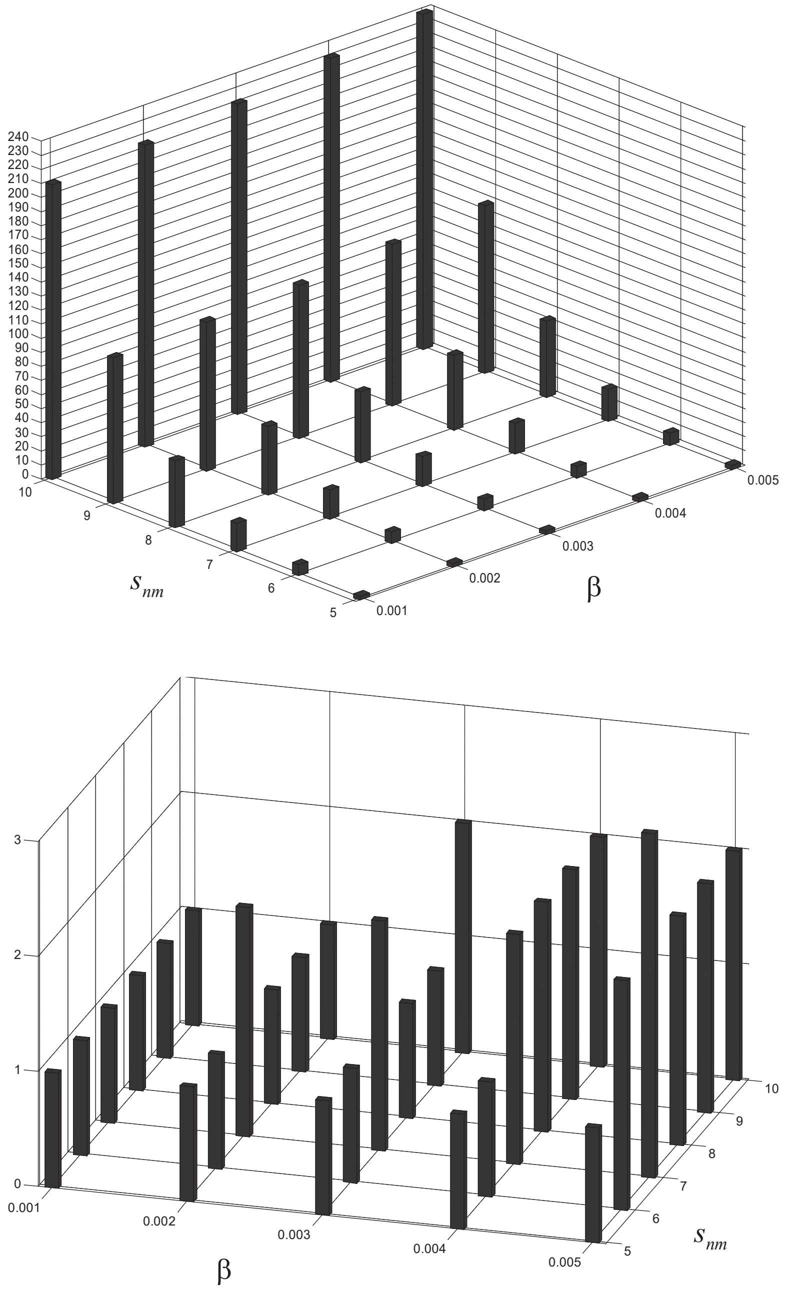
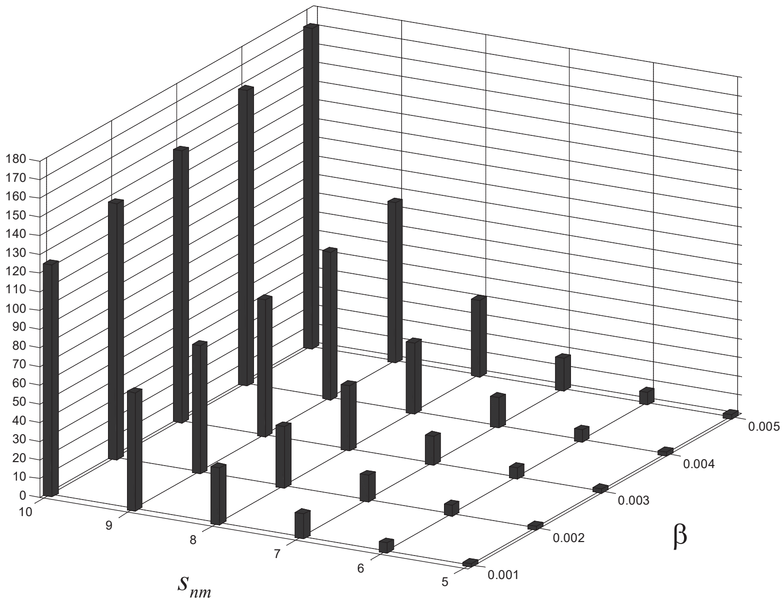


Figure 14. The longest time periods (in seconds) taken for finding $\{\beta\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations in a $3\times3\times3$ -FNCG

and their few cardinalities (the lower bar chart)
by the single $\{0.001\}_{n\in\{\overline{1,3}\}}$ -equilibrium situation



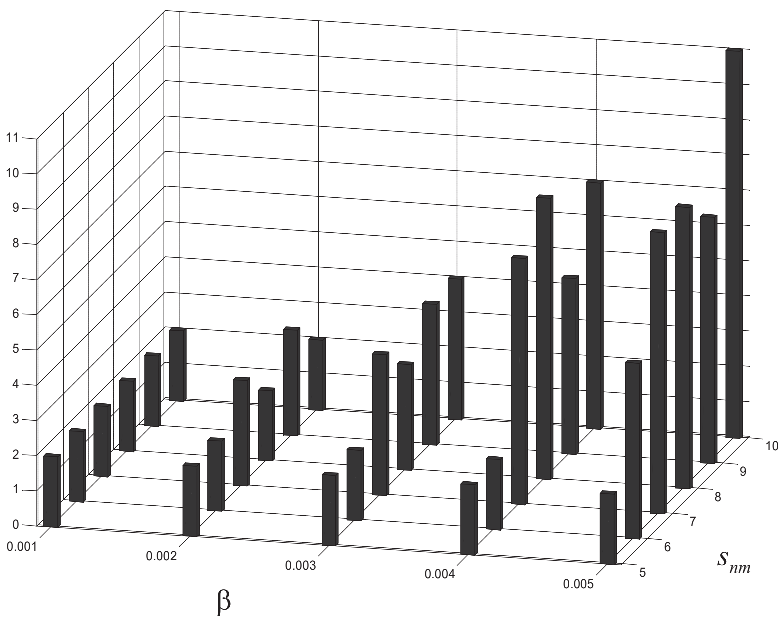


Figure 15. Time (in seconds) taken for finding $\{\beta\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations in a $3\times3\times3$ -FNCG and their cardinalities (the lower bar chart)

by the two $\{0.001\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations

and 11 $\{0.005\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations for $s_{nm} = 10$

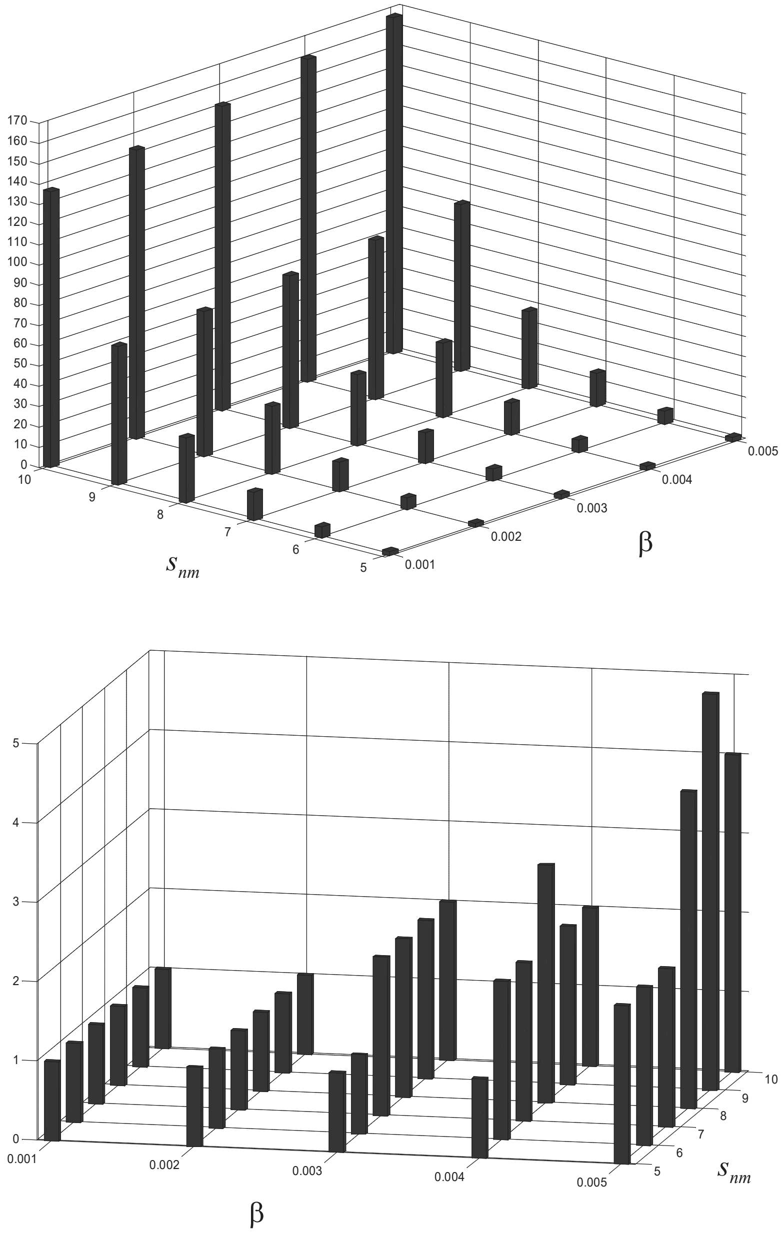


Figure 16. Time (in seconds) taken for finding $\{\beta\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations in a $3\times3\times3$ -FNCG and their cardinalities (the lower bar chart)

by the single $\{0.001\}_{n\in\{\overline{1,3}\}}$ -equilibrium situation

and the single $\{0.002\}_{n\in\{\overline{1,3}\}}$ -equilibrium situation

sampling step and increasing concessions lead to lengthening time periods (see the lengthened time in a $3\times3\times3$ -FNCG in Figure 17). The sampling step decrement causes abruptly increasing time periods for a $3\times3\times3\times3$ -FNCG, where $\{0.01\}_{n\in\{\overline{1,4}\}}$ -equilibrium situations are found occasionally in 3 minutes by $s_{nm}=5$, in 12 minutes by $s_{nm}=6$, in 42 minutes by $s_{nm}=7$, but it took 2 hours for $s_{nm}=8$. Adding a player worsens the timing: in a $\bigtimes_{r=1}^{5} 3$ -FNCG, $\{0.001\}_{n\in\{\overline{1,5}\}}$ -equilibrium situations are found occasionally in a half an hour by $s_{nm}=4$ and it took 3.2 hours for $s_{nm}=5$. Almost the same 3.2 hours are taken for finding $\{0.001\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations in a $4\times4\times4$ -FNCG by $s_{nm}=9$. An hour was taken for $s_{nm}=8$, and a half an hour was taken for $s_{nm}=7$. Only 5 minutes were taken for $s_{nm}=6$, and about 70 seconds were taken for $s_{nm}=5$ producing the probability 0.2 step.

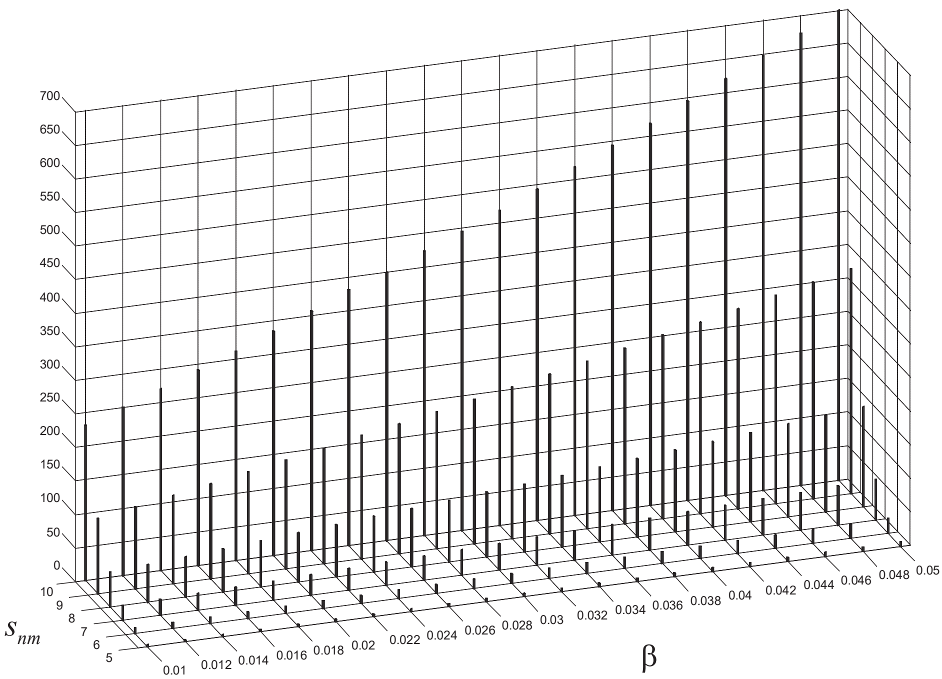


Figure 17. The lengthened time taken for finding $\{\beta\}_{n\in\{\overline{1,3}\}}$ -equilibrium situations in a $3\times3\times3$ -FNCG

Surely, integers $\left\{\left\{s_{nm}\right\}_{m=1}^{M_n-1}\right\}_{n=1}^{N}$ aren't to produce the probability $\left\{0.1z\right\}_{z\in\left\{\overline{1,\,5}\right\}}$ step necessarily. The step comes wider when the searching time is limited. And number of game cycles can't be always multiple of 5 or 10, so the wide step can be even $1/3$ or $1/2$.

Figure 18 shows time periods taken for finding $\left\{\beta\right\}_{n\in\left\{\overline{1,\,3}\right\}}$-equilibrium situations in a $3\times3\times3$-FNCG, when the sampling step is $1/7$ and wider through $1/4$. Cardinalities of those equilibria are unacceptably great. However, approximate Nash equilibrium situations by the probability $1/4$ step were found in a second (less than a second; the longest is 0.83 second). And it looks like this approximate equilibria time consumption by $\left\{\beta=0.001w\right\}_{w=1}^{5}$ is verified for any $3\times3\times3$-FNCG. Time periods taken for finding $\left\{\beta\right\}_{n\in\left\{\overline{1,\,4}\right\}}$-equilibrium situations in a $3\times3\times3\times3$-FNCG appeared to be about roughly 100 times longer (Figure 19). Here approximate Nash equilibrium situations by the least probability $1/4$ step were found in 37 seconds. A quick comparison of Figure 18 and Figure 19 reveals how the time grows much from a $3\times3\times3$-FNCG to a $3\times3\times3\times3$-FNCG. About a half an hour was taken for finding the single $\left\{0.001\right\}_{n\in\left\{\overline{1,\,4}\right\}}$-equilibrium situation by $s_{nm}=7$. Figure 20 shows almost ideal case of a $4\times4\times4$-FNCG, where the single $\left\{\left\{0.001w\right\}_{n\in\left\{\overline{1,\,3}\right\}}\right\}_{w=2}^{5}$-equilibrium situation is found only by $s_{nm}=7$ producing the probability $1/7$ step.

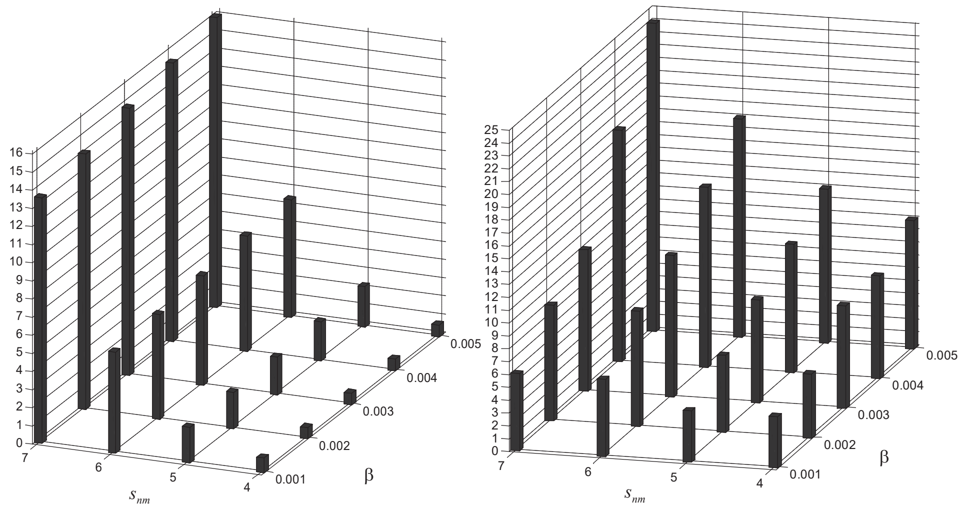


Figure 18. Time (in seconds) taken for finding $\left\{\beta\right\}_{n\in\left\{\overline{1,\,3}\right\}}$-equilibrium situations in a $3\times3\times3$-FNCG and their unacceptably great cardinalities (the right bar chart)

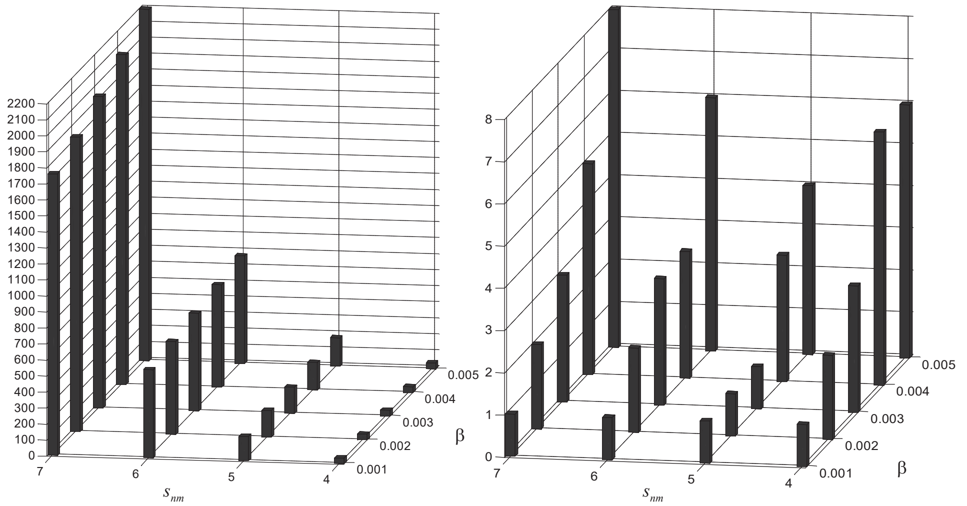


Figure 19. Time (in seconds) taken for finding $\{\beta\}_{n\in\{\overline{1,\,4}\}}$ -equilibrium situations in a $3\times3\times3\times3$ -FNCG and their cardinalities (the right bar chart)

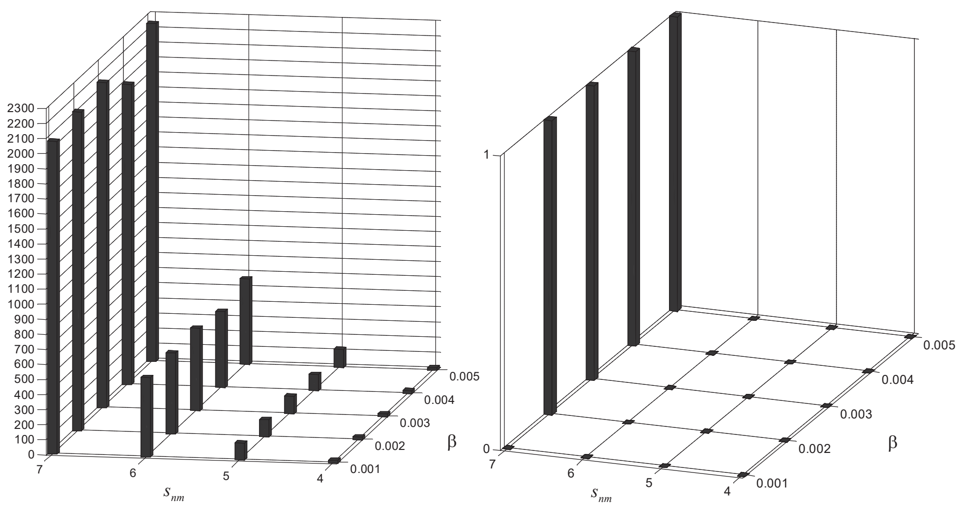


Figure 20. Time (in seconds) taken for finding the single $\left\{\{0.001w\}_{n\in\{\overline{1,\,3}\}}\right\}_{w=2}^{5}$ -equilibrium situation only by $s_{nm}=7$ in a $4\times4\times4$ -FNCG (the unit cardinality is on the right bar chart)

While trying to solve approximately a series of greater format FNCG, time periods which are spent for finding $\{\beta\}_{n\in\{\overline{1,\,6}\}}$ -equilibrium situations in $\bigtimes_{r=1}^{6} 3$-FNCG, $\{\beta\}_{n\in\{\overline{1,4}\}}$-equilibrium situations in $4\times4\times4\times4$-FNCG, $\{\beta\}_{n\in\{\overline{1,5}\}}$-equilibrium situations in $\bigtimes_{r=1}^{5} 4$-FNCG, and $\{\beta\}_{n\in\{\overline{1,6}\}}$-equilibrium situations in $\bigtimes_{r=1}^{6} 4$-FNCG are too long for $s_{nm}=5$ or for even $s_{nm}=4$. Nonetheless producing the probability 0.25 step herein, the 0.25-approximate equilibria wouldn't be fruitless. Obviously, the contrivance of routines (Figure 2 and Figure 3) for finding approximate Nash equilibrium situations runs out because of deep nested loops for those four FNCG hereinbefore exampled. And a new problem is how to accelerate finding approximate equilibria beyond the routine loop breaking.

12. Acceleration of finding approximate equilibria

Figures 4 — 20 prompt that, the greater concessions $\{\beta_n\}_{n\in B}$ or $\{\gamma_C\}_{C\subset\{\overline{1,N}\}}$ are, the more approximate equilibria we obtain. For effective practicing, the best case is when there is a single equilibrium situation or, sometimes, a few ones. The reason is we don't need additional choice problem [23]. Hence, to accelerate finding approximate equilibria, concessions $\{\beta_n\}_{n\in B}$ or $\{\gamma_C\}_{C\subset\{\overline{1,N}\}}$ should be assigned small. If set of $\{\beta\}_{n\in\{\overline{1,N}\}}$-equilibrium situations or strong $\{\gamma_C\}_{C\subset\{\overline{1,N}\}}$-equilibrium situations turns out empty, the failed concessions are increased at a small step.

When the expected payoff (4) is calculated, parallelization of matrix multiplication [24], [25], [26] can accelerate [27] the routine for finding approximate Nash equilibrium situations. Besides, the player's payoffs may be calculated on an individual processor core [28], [29], [30].

Adjustment of magnitudes $\{\alpha_n\}_{n=1}^{N}$ is subtler. If the n-th player's payoff variation restriction (11) at distance (10) for (8) and (9) is unfeasible, then either α_n is to be increased or sampling steps along simplex (3) dimensions are to be decreased. Any decrement of sampling steps leads to both the routine for building lattices and routine for finding approximate Nash equilibrium situations are slowed down. Therefore, magnitudes $\{\alpha_n\}_{n=1}^{N}$ primarily are counseled to be assigned great. Subsequently they may be decreased.

13. Discussion and conclusion

Whatever method of solving FNCG (1) is used, possible concessions arise always if Nash equilibria are not found. Of course, it concerns other types of equilibria or utility. Another motive of conceding payoffs is the DADP limitation.

Assigning values $\{\beta_n\}_{n\in B}$ or $\{\gamma_C\}_{C\subset\{\overline{1,\,N}\}}$ rationally allows to solve FNCG (1) much faster. The solution is implied as the single $\{\beta\}_{n\in\{\overline{1,\,N}\}}$-equilibrium situation or strong $\{\gamma_C\}_{C\subset\{\overline{1,\,N}\}}$-equilibrium situation. Two or three approximate equilibria are desirable rarer, except there is a risk of an approximate equilibrium situation appears disadvantageous to a player. A demerit is the rationale for $\{\beta_n\}_{n\in B}$ and $\{\gamma_C\}_{C\subset\{\overline{1,\,N}\}}$ is merely heuristic.

Selection of the sampling steps $\{s_{nm}^{-1}\}_{m=1}^{M_n-1}$ or the integers $\{s_{nm}\}_{m=1}^{M_n-1}$ is ruled by the restrictions imposed on them. Unfortunately, the n-th player's payoff variation restriction (11) at distance (10) for (8) and (9) by $q=\overline{1,\,N}$ depends utterly on how magnitudes $\{\alpha_n\}_{n=1}^{N}$ have been assigned before. Assignment of $\{\alpha_n\}_{n=1}^{N}$ is a preceding heuristics.

The version of routine for building lattices in Figure 1 is scarcely unique. But it is not worth to rationalize it — the routine is exercised very rapid. Routines for finding approximate Nash equilibrium situations in Figure 2 and Figure 3 might be optimized, though.

Nevertheless, mapping fundamental simplexes as sets of players' mixed strategies into lattices is followed by the eight plain merits:

1. The introduced fundamental simplexes' sampling allows to solve approximately any FNCG.

2. Owing to the sets of players' mixed strategies in FNCG are finitely sampled, the solution is practiced effectively, i. e. the player's payoff average in the solution situation converges to its expected payoff in this situation (due to that, by the proper number of game cycles, statistical frequencies approximate enough to support probabilities).

3. Number of approximate solutions is regulated by assigning values $\{\beta_n\}_{n\in B}$ or $\{\gamma_C\}_{C\subset\{\overline{1,\,N}\}}$ rationally. This also brings speedup in finding those solutions.

4. Owing to the DADP limitation, the payoff average convergence is rapid needing less game cycles (again, due to statistical frequencies approximate closer to support probabilities). Eventually, the solution or an arbitrary situation becomes applicable.

5. Sampling individually the player's fundamental simplex grants capability to manipulate pure strategies of various ranks. Then, the player samples dimensions of higher ranks with lesser steps, and dimensions of lower ranks are sampled sparser.

6. The routines are programmable within any environment. Priority environments are those who are CUDA enhanced [31], [32], [33] supporting multithreading modes [34], [35]. Special mathematical libraries are unnecessary.

7. The problem of unique solution is removable by adjusting concessions $\{\beta_n\}_{n\in B}$ or $\{\gamma_C\}_{C\subset\{\overline{1,\,N}\}}$.

8. The nested loop routines in Figure 2 and Figure 3 are easily retargeted on other types of equilibria or utility.

The work progression could be focused on the following unclear items:

1. Shall number of game cycles, DADP, and concessions be theoretically bound?

2. Does a maximal sampling step (for fully regular lattice having identical distance between its nodes) exist such that sampling steps mustn't be increased up from this maximum or else solutions become very different every time when sampling steps are changed?

3. Does a minimal sampling step (fully regular lattice) exist such that further decrement down from this minimum gives only similar (close) solutions?

4. Is there any possibility to determine ranges of sampling steps within which a number of approximate Nash equilibrium situations is constant?

These items, if ascertained, are believed to strengthen and supplement those eight merits. Proving theorems on convergence is supposed. But even without rigorous analysis, nonetheless, the suggested simplex finite approximation and concessions direct to solvability and applicability of any FNCG. And this is a basis in stating a universal approach for FNCG solution approximation in wide sense, where solvability, applicability, realizability, and adaptability would be unified.

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